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APPROXIMATIONS FOR THE DELIVERY SPLITTING MODEL

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ABSTRACT

Delivery splitting is a customer-handling policy aimed at reduction of the variance in the demand process, and hence at reduction of safety stocks. Customers with large demands are not delivered in one single batch, but in a number of equally spaced deliveries of size Q_{max} (except possibly for the last delivery), until the total demand is satisfied. It is supposed that for replenishment an (s, Q)-policy is used. In particular, in highly erratic demand processes such delivery splitting can be extremely useful. Two approximative methods are proposed to calculate for a given value of Q the minimal value of s that guarantees a prespecified service-level constraint. Validation of these computational methods is done by simulation, revealing case dependent performance for the first, but excellent performance for the second method. Finally a variant of the second method is proposed for situations in which knowledge about future deliveries (based on demands that have been split in the past) is explicitly used in the replenishment strategy. It is shown that this more advanced strategy may result in substantial stock level reductions on top of the reduction, that is achieved by delivery splitting itself.

Introduction

In a recent paper De Kok and Janssen (1995) explained how in a multi-stage distribution chain, including a manufacturer, sales-organisations and wholesalers, a rather stable demand pattern on the wholesalers side can be transformed into a highly unstable pattern for the manufacturer.

An explanation for this frequently occurring phenomenon is the rational (but short term) behaviour of two key persons in the distribution chain: the salesman of the sales organisation and the buyer on the wholesaler side respectively. A common objective for the salesman is to meet a turnover target each month. This objective is achieved by adaptive behaviour leading to high sales in the last week of the month stimulated by high discounts on certain products, but resulting in very low demand patterns in the first weeks of the next month. On the other hand most wholesalers act price-driven, which make them vulnerable to discount offers of salesmen. Indeed buyers will anticipate this behaviour and postpone orders towards the end of the month expecting on at least some of the products substantial discounts. This closes the circle and implies a more or less accepted negotiation process in which both salesman and buyer achieve their local i.e. personal objectives. This process induces high variations in monthly sales for individual products.

From this point of view it cannot be surprising that quite often in practice stocks are encountered which are up to three times as high as one would expect based on application of reorder point policies. It is noteworthy that those high inventories do not only occur on wholesalers side but also at the manufacturer, due to the highly erratic demand patterns faced by the manufacturer, which in turn require high safety stock levels to guarantee high fill rates. Of course the best way to break this vicious circle is to change the personal objectives of the managers involved, namely, to meet a turnover target by the salesman and to receive low prices by the buyer. However, this may be a long-term project and would require far reaching organizational changes.

For the short term an alternative and more operational procedure has been proposed through delivery splitting. In view of the above presented explanation for the erratic demand patterns at the manufacturer in which large demands are followed by periods of very low demand, it will be clear that the wholesaler is in general not interested to have a complete order delivered in one single batch. On the contrary, once the discount has been obtained through the large order, the wholesaler would prefer to have this order delivered in a number of disjoint deliveries according to a predetermined schedule of partial shipments. The first lot shipped is intended to replenish the wholesalers safety stock and the short term demand for the next couple of weeks. The remaining part of the order can then be shipped in a number of equal lots. This approach does not change the order quantity and the discount structure, but it affects the shipment flow from manufacturer to wholesaler, reducing the irregularity in the 'demand' patterns and reducing both the actual stock at the wholesaler and the safety stocks both at wholesaler and manufacturer.

The concept of delivery splitting is not new. In the literature the concept of delivery splitting is also known as order splitting which refers primarily to the possibilities to reduce lead times of replenishment orders by splitting these over more than one supplier (see e.g. Hong and Hayya (1992), Kelle and Silver (1990), Lau and Zhao (1993), Ramasesh et.al. (1991) and Sculli and Wu (1981)). Hong and Hayya (1992) give a chronological summary of the literature about order splitting. In the earliest papers (Kelle and Silver (1990) and Sculli and Wu (1981)) only stock reductions were considered. The increasing delivery frequency due to splitting of orders was not incorporated while demand was assumed to be constant. The papers of Hong and Hayya (1992) and Ramasesh et.al. (1991) consider total relevant costs functions, including costs of more frequent deliveries, but still only constant demand models where analysed. These papers stress the trade-off that has to be made between an increase in delivery costs on one hand and the decrease in holding costs when order splitting is applied. Lau and Zhao (1993) give a procedure for determining the optimal order policy in case of stochastic demand, i.e. the total order quantity, reorder point (subject to a maximum allowable stockout risk) and the proportion of split between two suppliers. They do not exclude the possibility to split a single order in two orders at the same supplier. The results of the papers about order splitting show a considerable improvement in the customers performance measures (mean and standard deviation of demand during the lead time, inventory on hand and number of backlogs) when split orders are used. Especially in Lau and Zhao (1993) it is shown that stock reductions depend primarily on the difference in the lead times of both suppliers that are involved.

In spite of the resemblences, however, the concept of order splitting differs from the concept of delivery splitting that we deal with. While order splitting primarialy refers to the division of one replenishment order over more than one supplier to reduce lead times, (and as such takes the customers point of view), delivery splitting refers to dividing a (large) customer order in equally sized deliveries by one and the same customer. The focus of delivery splitting is to reduce the variance in the demand process and to achieve stock reductions at the customers as a by-product.

De Kok and Janssen (1995) use simulation to provide an impression of the size of the effect of delivery splitting on inventory holding costs. These effects are balanced against increased transportation costs due to more frequent shipments. In particular in view of setting the proper level of the reorder point while using delivery splitting it would be advantageous to have an analytic approach for this model. Although the analysis given by De Kok and Janssen provides insights into the possibly large effects delivery splitting may have, it should be admitted that simulation is not the appropriate tool to use when the concept of delivery splitting is to be implemented in an operational setting. Apart from the usual decision variables (order quantity and reorder point) through delivery splitting at least two additional decision variables (the standard delivery quantity and the delivery interval) have to be set, which makes simulation less appropriate as optimization tool.

In this paper an analytic approach is presented which enables us to quantify the effects of delivery splitting on inventory costs and transportation costs. The underlying replenishment strategy that is used is a basic reorder point policy, characterised by two control parameters s and Q, where s denotes the reorder point and Q the reorder quantity.

However, in the context of delivery splitting it turns out to be crucial whether or not information about future deliveries is used when implementing an (s, Q)-policy. Namely when customer orders are split into multiple deliveries information becomes available about future deliveries. It is clear that using this information explicitly in the decision whether or not to replenish improves the performance of the replenishment strategy. In other words it makes a substantial difference whether the inventory is controlled based upon the inventory position (actual stock level plus outstanding orders minus backlogs) or on the 'free inventory position', which is defined as the inventory position minus planned future deliveries within the lead time (due to splitting of previously placed orders).

The organisation of this paper is as follows. In section 1 we present a detailed description of the delivery splitting model. In sections 2 and 3 two methods are presented to calculate the optimal reorder level s in a situation where delivery splitting is used, given a P_2-service level (i.e. the long-run fraction of demand delivered directly from shelf) and given the reorder quantity Q. The method of section 2 is 'quick and dirty', while the method of section 3 is more accurate and sophisticated at the expense of increased complexity and computational requirements. In section 4 we will present an alternative model where information about future deliveries (due to splitting of previously placed orders) is used explicitly in the sense that order decisions are based on 'free inventory position' instead of the inventory position itself. In section 5 we present a numerical analysis aimed at validation of the approximative analytical approaches of sections 2, 3 and 4. The comparisons are made through simulation. Also we present an indication of the effects of using delivery splitting and using information about the future deliveries in controlling the system. Finally in section 6 we present some conclusions and recommendations for future research. Technical details of the analysis are given in eight appendices. Appendices 9,10 and 11 refer to the numerical experiments.

1. Model description

In an inventory model we distinguish three defining components: the demand process, the replenishment policy and the delivery rules. The description of each of these components for the model with delivery splitting is given below.

We assume that the demand process is a compound Poisson process with arrival rate λ and a probability distribution function of the demand size $F_D(.)$ with density function $f_D(.)$. As stated before delivery splitting seems to be of particular interest in case of a demand process where small demands are interspersed with very large demands. A generalized Erlang distribution is suitable to model this type of demand process. In our numerical examples we will therefore use

$$f_D(t) \;=\; p\mu_1^k \frac{t^{k-1}}{(k-1)!}e^{-\mu_1 t} + (1-p)\mu_2^l \frac{t^{l-1}}{(l-1)!}e^{-\mu_2 t} \quad (t \geq 0) \tag{1.1}$$

Replenishment of stock occurs according to a continuous review (s,Q)-policy, i.e. as soon as the inventory position drops below the reorder point s a multiple of Q is ordered, such that the inventory position after ordering is between s and $s+Q$. As a service performance measure the P_2-service measure (often denoted as the fill rate) is used: the long-run fraction of demand delivered directly from shelf, see e.g. Silver and Peterson (1985), Tijms and Groenevelt (1984). The lead time, L, of the replenishment orders is supposed to be deterministic. This guarantees that replenishment orders do not cross in time.

Orders which cannot be delivered directly from stock on hand will be backordered. However large demands will not be delivered in one single batch, even in case the inventory level is sufficiently large. The customer receives only a limited quantity, Q_{max}, at a time. If the demand size is larger than Q_{max}, starting at the demand epoch, an amount of Q_{max} is delivered in a number of shipments which are T time units apart . Consequently all quantities delivered are equal to Q_{max} except possibly the last. When a customer with demand of size D arrives at epoch t results in the following delivery scheme:

$$\begin{cases} \text{deliver } \; Q_{max} \; \text{ on epoch } \quad t+jT & \qquad 0 \leq j < n \\ \text{deliver } \; D - nQ_{max} \; \text{ on epoch } \quad t+nT & \end{cases} \tag{1.2}$$

where $n := \max\{m \in I\!N | mQ_{max} \leq D\}$.

The problem now is to find, for given values of Q, Q_{max} and T, the minimal value of the reorder point s for which a predetermined P_2-service level β is achieved. It is intuitively clear that the minimal value of the reorder point s will increase with increasing values of Q_{max}. The case in which delivery splitting is not allowed can be identified as the situation where Q_{max} is equal to ∞.

In an operational setting of this problem also Q_{max} and T can be considered as decision variables. To find optimal values for Q_{max} and T one has to balance a decrease in inventory holding costs (Q_{max} small) against increased transportation costs. Moreover, the choice of T should be related to the lead time L of a replenishment order (see also the numerical experiments in section 5).

To close this section we recall a general relationship between the minimal reorder level s and the required service level β in case of no delivery splitting (we refer to De Kok (1991) for a complete analysis of this relationship) Denote by $Z(L)$ the total demand during the lead time L and by U the undershoot under the reorder point s of the demand that triggers a replenishment. Then it is known that s and β are related through

$$1-\beta=\begin{cases} 1 & s \leq -Q \\ \frac{EZ-s-E(Z-(s+Q))^+}{Q} & -Q < s \leq 0 \\ \frac{E(Z-s)^+ - E(Z-(s+Q))^+}{Q} & 0 < s \end{cases} \tag{1.3}$$

where $Z = Z(L) + U$ and $X^+ = \max(0, X)$.
From (1.3) the reorder point s can be calculated according to the method presented by Tijms and Groenevelt (1984), where it is assumed that the distribution of Z can be approximated by a generalized Erlang distribution (see Appendix 1).

2. A fast approximative method for calculating the reorder point for the (s, Q)-policy with delivery splitting

In this and the following section we describe two approximative methods to calculate the reorder point in case delivery splitting is applied but information about future deliveries is not taken into account explicitly. The difference between this model and the standard (s, Q) model is caused by the way customer orders are delivered. In the standard (s, Q) model the delivery process and the demand process coincide when the physical stock is sufficiently large. Thus, according to assumptions made in section 1, the delivery process is a compound Poisson process with rate λ and distribution function F_D. Delivery splitting affects the delivery process in the sense that one single demand gives now rise to a sequence of smaller sized and equally spaced deliveries, even when the physical stock is sufficiently large (see (1.2)).

In general it will be very difficult to exactly describe the resulting stochastic process of delivery occurrences. So approximative approaches seem to be necessary. In this section we make the following simplifying assumptions. The stochastic process of delivery occurrences is considered to be a superposition of an (in principal) infinite number of independent compound Poisson processes. The i-th process in this sequence can be considered as the i-th generation demand offspring, i.e. for $i = 0, 1, \ldots$ a delivery occurs at time t in the i-th offspring process if and only if at $t - iT$ an original demand occurred of size larger than iQ_{max}. The assumption that the sequence of compound Poisson processes constitutes a sequence of *independent* processes is the simplification that is made here. It is intuitively clear that the validity of this assumption will improve with increasing values of T.

In determining the minimal value of the reorder point s to satisfy the fill rate constraint, we first conclude that formula (1.3) remains valid for the case of delivery splitting. The only difference is that the demand process has been changed according to the explanation given above. To determine explicit expressions for the right hand side of (1.3) we follow the procedure proposed by Tijms and Groenevelt (1984), i.e. we approximate the distribution of Z by an generalized Erlang distribution of the type (1.1). For application of (1.3) it is sufficient to have explicit expressions for the first two moments of Z. This in turn requires knowledge of the first three moments of the demand sizes in the superposed delivery process. Details are presented below, while for the derivation of various formulas we refer to appendix 2.

First we introduce some notation, where quantities that refer to the superposition of the offspring processes are indicated with *,

λ^*	:=	delivery intensity (in the superposition of offspring processes);
D_j^*	:=	size of j-th delivery in the superposed process ($j = 1, 2, \ldots$);
$N^*(t)$	:=	total number of deliveries in $(0, t]$;
$Z^*(t)$	:=	total amount delivered in $(0, t]$;
U^*	:=	the undershoot under the level s of the superposed delivery process;
λ_i	:=	delivery intensity of the i-th offspring process ($i = 0, 1, \ldots$);
$D_{i,j}$	:=	size of j-th delivery generated ($j = 1, 2, \ldots$) by the i-th offspring process ($i = 0, 1, \ldots$);
$N_i(t)$	:=	number of deliveries in $(0, t]$ generated by the i-th offspring process ($i = 0, 1, \ldots$);
$Z_i(t)$	:=	total amount delivered in $(0, t]$ due to deliveries generated by i-th offspring process ($i = 0, 1, \ldots$).

Now formula (1.3) has to be solved while replacing $Z(L)$ and U by $Z^*(L)$ and U^* respectively. The following relations hold (see Appendix 2)

$$Z^*(t) = \sum_{i=0}^{\infty} Z_i(t) \tag{2.1}$$

$$\lambda^* = \sum_{i=0}^{\infty} \lambda_i \tag{2.2}$$

$$I\!ED_1^{*n} = \sum_{i=0}^{\infty} \frac{\lambda_i}{\lambda^*} I\!ED_{i,1}^n \qquad (n = 1, 2, \ldots) \tag{2.3}$$

The appropriate choice for λ_i is given by

$$\lambda_i = \lambda(1 - F_D(iQ_{max})) \qquad (i = 0, 1, \ldots) \tag{2.4}$$

After straightforward analysis we can rewrite (2.3) for $n = 1, 2$ into (see Appendix 3):

$$I\!ED_1^* = \frac{\lambda}{\lambda^*} I\!ED \tag{2.5}$$

$$I\!ED_1^{*2} = \frac{\lambda}{\lambda^*}\left(I\!ED^2 - 2Q_{max}\sum_{i=1}^{\infty}\int_{iQ_{max}}^{\infty}(1 - F_D(x))dx\right) \tag{2.6}$$

Note that $I\!ED_1^{*2} \leq I\!ED^2$, which indicates the reduction in variation. In case of a generalized Erlang distribution for the demand sizes, (2.3) can be computed explicitly (see Appendix 4). Because L is deterministic and $N^*(t)$ is assumed to be a Poisson process we find, using (2.5), for the total amount delivered during the lead time

$$\begin{aligned} I\!EZ^*(L) &= I\!EN^*(L) I\!ED_1^* \\ &= \lambda^* L\, I\!ED_1^* \\ &= \lambda L\, I\!ED \end{aligned} \tag{2.7}$$

$$\begin{aligned} \sigma^2(Z^*(L)) &= I\!EN^*(L)\sigma^2(D_1^*) + \sigma^2(N^*(L)) I\!ED_1^{*2} \\ &= \lambda^* L \sigma^2(D_1^*) + \lambda^* L I\!ED_1^{*2} \end{aligned} \tag{2.8}$$

Now using that the distribution of the undershoot has approximately a residual lifetime distribution and using results from renewal theory gives (see Tijms (1994), pp. 10):

$$I\!EU^* \simeq \frac{I\!ED_1^{*2}}{2 I\!ED_1^*} \tag{2.9}$$

$$\mathbb{E}U^{*2} \simeq \frac{\mathbb{E}D_1^{*3}}{3\mathbb{E}D_1^{*}} \tag{2.10}$$

Combining (2.7) to (2.10) with (2.5) and (2.6) we have expressions for the first two moments of $Z^*(L)$ and U^*. Fitting the distribution of Z by an generalized Erlang distribution enables us to calculate the reorder point s for given values of Q, Q_{max} and β with (1.3).

It is worthwhile to note that the above described approximation procedure leads to values of s which are independent of the actual value of T.

In section 5 we evaluate the validity of this approximate procedure.

3. A more advanced reorder point calculation.

In this section we present a more advanced method for the calculation of the reorder point s, given the values of the other parameters β, Q, Q_{max} and T. The difference with the method presented in the previous section is that here the correlation structure between subsequent deliveries is taken into account. The approach is again based on the application of formula (1.3), where again we assume that the distribution of Z can be approximated by a generalized Erlang distribution, which implies that for application of (1.3) we only need the first two moments of $Z(L)$ (the total amount delivered during the lead time L minus the deliveries due to the particular customer who triggered the replenishment) and U (the undershoot under the reorder point of the delivery that triggers the replenishment plus the remaining deliveries during the lead time of the associated customer). Assuming that an order is triggered at time t we focus on the delivery process during the interval $[t, t+L)$ and note that $Z(L)$ can be decomposed into two parts:

$Z_1(L)$:= the total amount delivered during the lead time due to new customers arriving during the lead time;

$Z_2(L)$:= the total amount delivered during the lead time due to customers who arrived before t,

and

U := the undershoot of the delivery that triggers the replenishment plus the remaining deliveries during the lead time of the associated customer.

The random variable Z occurring in (1.3) in now given by

$$Z = Z(L) + U = Z_1(L) + Z_2(L) + U \tag{3.1}$$

To illustrate the influence of various customers on the total amount to be delivered during the lead time L we introduce the function $m(\tau)$ denoting the maximum number of deliveries during an interval of length L caused by a customer who arrived at time τ ($\tau \in I\!R$) when the replenishment was triggered at time 0. In determining an expression for $m(\tau)$ we agree that a replenishment is handled before a delivery when they coincide in time, which occurs with positive probability when L in an integral multiple of T, (see also remark 3.1 at the end of this section). Now a moment reflection reveals (see Figure 3.1) that

$$m(\tau) = \begin{cases} 0 & \tau \geq L & \\ i & L - iT \leq \tau < L - (i-1)T & i = 1, \ldots, \lfloor \frac{L}{T} \rfloor \\ \lfloor \frac{L}{T} \rfloor & \xi - iT \leq \tau < -(i-1)T & i = 1, 2, \ldots \\ \lfloor \frac{L}{T} \rfloor + 1 & -iT \leq \tau < \xi - iT & i = 0, 1, \ldots \end{cases} \tag{3.2}$$

where $\lfloor x \rfloor := \max\{n \leq x | n \in I\!N\}$ and $\xi := L - \lfloor \frac{L}{T} \rfloor T$.

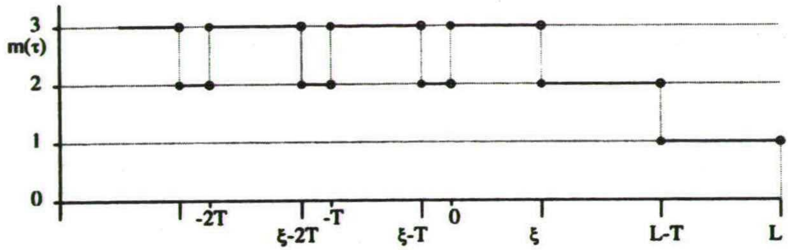

Figure 3.1 The function $m(\tau)$ $(T = 4; L = 11)$

Next we define

$D^{(1)}_{i,j}$:= the contribution to $Z_1(L)$ of the j-th customer arriving in interval $[L - iT, L - (i-1)T)$, $i = 1, \ldots, \lfloor \frac{L}{T} \rfloor$, $j = 1, 2, \ldots$;

$N^{(1)}_i$:= number of customers that arrive during the interval $[L - iT, L - (i-1)T)$, $i = 1, 2, \ldots, \lfloor \frac{L}{T} \rfloor$;

$D^{(1)}_{\lfloor \frac{L}{T} \rfloor + 1, j}$:= the contribution to $Z_1(L)$ of the j-th customer arriving during the interval $[0, \xi)$, $j = 1, 2, \ldots$;

$N^{(1)}_{\lfloor \frac{L}{T} \rfloor + 1}$:= number of customers that arrive during the interval $[0, \xi)$;

$D^{(2)}_{i,j,A}$:= the contribution to $Z_2(L)$ of the j-th customer arriving during the interval $[\xi - iT, -(i-1)T)$, $i = 1, 2, \ldots,$ $j = 1, 2, \ldots$;

$N^{(2)}_{i,A}$:= number of customers that arrive during the interval $[\xi - iT, -(i-1)T)$, $i = 1, 2, \ldots$;

$D^{(2)}_{i,j,B}$:= the contribution to $Z_2(L)$ of the j-th customer arriving during the interval $[-iT, \xi - iT)$, $i = 1, 2, \ldots$; $j = 1, 2, \ldots$;

$N^{(2)}_{i,B}$:= number of customers that arrive during the interval $[-iT, \xi - iT)$, $i = 1, 2, \ldots$.

Then we have

$$Z_1(L) = \sum_{i=1}^{\lfloor \frac{L}{T} \rfloor + 1} \sum_{j=1}^{N^{(1)}_i} D^{(1)}_{i,j} \tag{3.3}$$

$$Z_2(L) = \sum_{i=1}^{\infty} \Big(\sum_{j=1}^{N^{(2)}_{i,A}} D^{(2)}_{i,j,A} + \sum_{j=1}^{N^{(2)}_{i,B}} D^{(2)}_{i,j,B} \Big) \tag{3.4}$$

First note that all contributions to $Z_1(L)$ and $Z_2(L)$ (i.e. $D^{(1)}_{i,j}$, $D^{(2)}_{i,j,A}$ and $D^{(2)}_{i,j,B}$) are mutually independent because the demands of different customers are independent of each other.

Next we define for some generic random variable D with distribution function F_D

$$\bar{D}_{k,l} := \min\{(D - kQ_{max})^+, lQ_{max}\} \qquad (k, l = 0, 1, \ldots) \tag{3.5}$$

and we note that ($\stackrel{d}{=}$ denotes equality in distribution)

$$D^{(2)}_{i,j,A} \stackrel{d}{=} \bar{D}_{i,\lfloor \frac{L}{T} \rfloor} \qquad (i,j = 1,2,\ldots) \tag{3.6}$$

and

$$D^{(2)}_{i,j,B} \stackrel{d}{=} \bar{D}_{i,\lfloor \frac{L}{T} \rfloor + 1} \qquad (i,j = 1,2,\ldots) \tag{3.7}$$

while

$$D^{(1)}_{i,j} \stackrel{d}{=} \bar{D}_{0,i} \qquad (i = 1,2,\ldots,\lfloor \frac{L}{T} \rfloor + 1\,;\, j = 1,2,\ldots) \tag{3.8}$$

It easily follows that the n-th moment $(n = 1,2,\ldots)$ of $\bar{D}_{k,l}$ $(k = 0,1,\ldots, l = 1,2,\ldots)$ satisfies

$$\mathbb{E}(\bar{D}_{k,l})^n = \int_{kQ_{max}}^{(k+l)Q_{max}} (x - kQ_{max})^n dF_D(x) + (lQ_{max})^n (1 - F_D((k+l)Q_{max})) \tag{3.9}$$

and (see Appendix 5)

$$\bar{D}_{k,l} = \sum_{j=k}^{k+l-1} \bar{D}_{j,1} \qquad (k = 0,1,\ldots; l = 1,2,\ldots) \tag{3.10}$$

From (3.1) to (3.8) we conclude that (see also appendix 6)

$$\begin{aligned}
\mathbb{E}Z(L) &= \sum_{i=1}^{\lfloor \frac{L}{T} \rfloor} \lambda T\, \mathbb{E}\bar{D}_{0,i} + (\lambda\xi)\mathbb{E}\bar{D}_{0,\lfloor \frac{L}{T} \rfloor + 1} \\
&\quad + \sum_{i=1}^{\infty} \left(\lambda(T-\xi)\mathbb{E}\bar{D}_{i,\lfloor \frac{L}{T} \rfloor} + (\lambda\xi)\mathbb{E}\bar{D}_{i,\lfloor \frac{L}{T} \rfloor + 1} \right) \\
&= \lambda L\, \mathbb{E}D && (3.11) \\
\sigma^2(Z(L)) &= \sum_{i=1}^{\lfloor \frac{L}{T} \rfloor} \lambda T\, \mathbb{E}\bar{D}^2_{0,i} + (\lambda\xi)\mathbb{E}\bar{D}^2_{0,\lfloor \frac{L}{T} \rfloor + 1} && (3.12) \\
&\quad + \sum_{i=1}^{\infty} \left(\lambda(T-\xi)\mathbb{E}(\bar{D}^2_{i,\lfloor \frac{L}{T} \rfloor}) + (\lambda\xi)\mathbb{E}\bar{D}^2_{i,\lfloor \frac{L}{T} \rfloor + 1} \right)
\end{aligned}$$

Notice that (3.11) is equal to (2.7), but (3.12) differs from (2.8) in the sense that in (3.12) correlations between the offspring processes are taken into account.

To apply the reorder point formula (1.3), using (3.11) and (3.12), we still need explicit expressions for the first two moments of U. To obtain approximations for the first two moments of U we again follow the approach of section 2, where the total delivery process is interpreted as a superposition of independent compound Poisson processes (the so-called offspring processes). Note that there still remains a substantial difference with section 2 since here the approach with the offspring processes is only used to compute approximations for U, while in section 2 it was used for the computation of the first two moments of both $Z(L)$ and U.

We define:

$\hat{D}_i$:= the remaining amount to be delivered to the customer who triggered a replenishment through the i-th offspring process (including the i-th offspring delivery) $(i = 0, 1, \ldots)$:

$\hat{D}_{i,k}$:= $\min\{\hat{D}_i, kQ_{max}\}$, $(i = 0, 1, \ldots)$;

U_i := the total contribution to U of the customer who triggered a replenishment through the i-th offspring process $(i = 0, 1, \ldots)$;

q_i := the probability that a replenishment is triggered through the i-th offspring process $(i = 0, 1, \ldots)$.

Using renewal type arguments it can be shown that q_i can be approximated by the following formula

$$q_i \simeq \frac{\lambda_i \mathbb{E} D_{i,1}}{\sum_{j=0}^{\infty} \lambda_j \mathbb{E} D_{j,1}} = \frac{\mathbb{E} \bar{D}_{i,1}}{\mathbb{E} D} \qquad (i = 0, 1, \ldots) \tag{3.13}$$

provided $Q - Q_{max}$ is not too small (see Appendix 7 for the formal proof of this result).

Next we derive an approximate probability distribution of $\hat{D}_i$. For this purpose we note that in a standard inventory control process, where demands are generated by a sequence of i.i.d. random variables $(W_j)_{j=1}^{\infty}$, with density function $f_W(x)$, the density function $f_{\hat{W}}(x)$ of the demand $\hat{W}$ that causes the undershoot under the reorder point s can be approximated by (see e.g. Tijms (1994) pp. 85 or Cox (1962) pp. 65-66)

$$f_{\hat{W}}(w) \simeq \frac{w f_W(w)}{\mathbb{E} W_1} \tag{3.14}$$

Denote the inventory position just prior to a replenishment by V. Then it can be shown that V given $\hat{W} = w$ is uniformly distributed over $(s, s+w)$.

Now note that in the situation under consideration all actual deliveries (which are also 'trigger'-quantities) are truncated to Q_{max}, while the actual value of $\hat{D}_i$ can be larger than Q_{max}. Thus realizations of $\hat{D}_i$ bigger than Q_{max} have equal probability to trigger, where the probability of triggering a replenishment for realizations of $\hat{D}_i$ smaller than Q_{max} are proportional to their actual size (according to (3.14)). Using the appropriate analogue of formula (3.14), the density function of $\hat{D}_i$ can be approximated by

$$f_{\hat{D}_i}(x) \simeq \frac{\min\{x, Q_{max}\} f_D(x + iQ_{max})}{\mathbb{E} \bar{D}_{i,1}} \qquad (i = 0, 1, \ldots) \tag{3.15}$$

Define

V_i := the inventory position just prior to a replenishment when a replenishment is triggered through the i-th off spring process $(i = 0, 1, \ldots)$.

Then using the appropriate analogue of V we conclude that V_i given $\hat{D}_i = d$ is uniformly distributed over $(s, s + \min\{d, Q_{max}\})$.

Also note that the remaining amount to be delivered within the lead time is at most $\lceil \frac{L}{T} \rceil Q_{max}$, where $\lceil x \rceil := \min\{n \geq x | n \in \mathbb{N}\}$
Hence

$$U_i = \hat{D}_{i, \lceil \frac{L}{T} \rceil} - V_i \qquad (i = 0, 1, \ldots) \tag{3.16}$$

From (3.16) we find, after some straightforward calculations, the following expressions for the first two moments of U (see Appendix 8)

$$\mathbb{E}U \simeq \frac{\mathbb{E}D_1^{*2}}{2\mathbb{E}D_1^*} + \frac{Q_{max}}{\mathbb{E}D}\sum_{i=0}^{\infty}\mathbb{E}\bar{D}_{(i+1),\lceil\frac{L}{T}\rceil-1} \tag{3.17}$$

$$\mathbb{E}U^2 \simeq \frac{\mathbb{E}D_1^{*3}}{3\mathbb{E}D_1^*} + \frac{Q_{max}}{\mathbb{E}D}\sum_{i=0}^{\infty}\mathbb{E}(\bar{D}_{(i+1),\lceil\frac{L}{T}\rceil-1})^2 + \frac{Q_{max}^2}{\mathbb{E}D}\sum_{i=0}^{\infty}\mathbb{E}\bar{D}_{(i+1),\lceil\frac{L}{T}\rceil-1} \tag{3.18}$$

Again fitting a generalized Erlang distribution to the distribution of Z and using (3.11), (3.12), (3.17), and (3.18) enables us to compute the optimal reorder point s from (1.3).

Remark 3.1.
It can be shown that in case $T \geq L$ formula (3.11) reduces to (2.8) and moreover the expressions (3.17) and (3.18) for the first two moments of the undershoot reduce to the corresponding expressions (2.9) and (2.10) in section 2. This implies that in case $T \geq L$ in fact the methods of section 2 and 3 give the same results.

Remark 3.2.
In case we agree that a delivery is handled before a replenishment order when they coincide in time, then relation (3.11) and (3.12) still hold. However, the remaining amount to be delivered within the lead time of the customer that triggers the replenishment is maximal $(\lfloor\frac{L}{T}\rfloor + 1)Q_{max}$ (instead of $\lceil\frac{L}{T}\rceil Q_{max}$). Thus $U_i = \hat{D}_{i,\lfloor\frac{L}{T}\rfloor+1} - V_i$ $(i = 0, 1, \ldots)$ and therefore (3.17) and (3.18) have to be adapted straightforwardly. Also note that the method in section 2 is invariant for the priority rule for replenishment orders and deliveries.

4. A replenishment strategy based on known future deliveries

So far we considered a replenishment policy which is only based on the inventory position i.e. physical inventory level plus stock on order replenishments minus backorders. However, in case of delivery splitting there exists explicit knowledge about the occurrence of future deliveries of splitted orders. This knowledge could be used to improve the performance of the inventory system.

In this section we will deal with an inventory replenishment policy of (s, Q)-type which is not based on the inventory position at time t but on the inventory position at time t minus all planned future deliveries in $(t, t+L]$. This actually resembles the 'available to promise' inventory level as used in MRP-systems, although the 'available to promise' concept in the MRP-context also takes into account the timing of both stock replenishment and customer orders.

We use the following notation:

$I(t)$:= inventory position at time t;

$K(t)$:= the known deliveries during the interval $(t, t+L]$;

$H(t)$:= $I(t) - K(t)$.

In this section the (s, Q) policy prescribes to place a replenishment order of size Q (or multiples of Q) as soon as $H(t)$ drops below the level s. As usual we denote by U, the undershoot under s, the difference between s and $H(t)$ immediately after a replenishment is triggered and $Z(L)$ denotes the ***unknown*** demand during a lead time L.

Again formula (1.3) is in order to calculate the optimal reorder level s. Application of this formula using the approximate method of Tijms and Groenevelt (1984) requires the first two moments of the independent random variables U and $Z(L)$. First we note that $Z(L)$ simply equals $Z_1(L)$ as defined in section 3.
Hence we conclude from (3.3) and (3.11) that

$$\mathbb{E}(Z(L)) = \sum_{i=1}^{\lfloor \frac{L}{T} \rfloor} \lambda T \mathbb{E}(\bar{D}_{0,i}) + (\lambda\xi)\mathbb{E}(\bar{D}_{0,\lfloor \frac{L}{T} \rfloor+1}) \tag{4.1}$$

$$\sigma^2(Z(L)) = \sum_{i=1}^{\lfloor \frac{L}{T} \rfloor} \lambda T \mathbb{E}(\bar{D}_{0,i}^2) + (\lambda\xi)\mathbb{E}(\bar{D}_{0,\lfloor \frac{L}{T} \rfloor+1}^2) \tag{4.2}$$

To derive the first two moments of U we have to examine the evolution of the process $\{H(t),\ t \geq 0\}$ more closely. Note that $H(t)$ is the difference of two processes $\{I(t),\ t \geq 0\}$ and $\{K(t),\ t \geq 0\}$. Before going into detail, we notice that one has to be careful in the special case where L is a integral multiple of T. Consider, for purpose of illustration, the case $T = L$. In this situation it may happen that two events will coincide in time, namely, the arrival of the replenishment order and the actual delivery that triggered the replenishment. When a replenishment order is handled before a delivery in case they coincide in time (as assumed in the previous sections) and the replenishment is triggered by a planned delivery that comes within view at the end of the lead time then this planned delivery itself does not contribute to the amount delivered during the lead time. For this reason we assume in this section that deliveries are handled before replenishment orders in case both coincide in time.

Also note that the maximal number of deliveries within the lead time of a new customer for all combinations of T and L is equal to $\lfloor \frac{L}{T} \rfloor + 1$.

At an arbitrary time epoch t there may occur two possible events which affect the inventory position $I(t)$.

- A new arriving customer at t with demand D decreases $I(t)$ with $\min\{D, Q_{max}\}$.
- A customer who arrived at epoch $t - nT\,(n = 1, 2, \ldots)$ with demand $D > nQ_{max}$ decreases $I(t)$ with $\min\{D - nQ_{max}, Q_{max}\}$.

At an arbitrary time epoch t three events may occur which affect $K(t)$

- A new arriving customer at epoch t with demand D increases $K(t)$ with $\min\{D - Q_{max}, \lfloor \frac{L}{T} \rfloor Q_{max}\}$.
- A customer who arrived at epoch $t - nT\,(n = 1, 2, \ldots)$ with demand $D > nQ_{max}$ decreases $K(t)$ with $\min\{D - nQ_{max}, Q_{max}\}$.
- A customer who arrived at epoch $t + L - nT\,(n = \lfloor \frac{L}{T} \rfloor + 1, \lfloor \frac{L}{T} \rfloor + 2, \ldots)$ with demand $D > nQ_{max}$ increases $K(t)$ with $\min\{D - nQ_{max}, Q_{max}\}$.

Note that the second effect on $K(t)$ is caused by a change from a planned delivery into an actual delivery, while the third effect is caused by a change from a planned delivery outside the lead time period L into a planned delivery inside the lead time period. Also we note that the second effect on $I(t)$ and the second effect on $K(t)$ neutralize each other as far as $H(t)$ is concerned. Hence $H(t)$ is affected by only two events

- A new arriving customer at epoch t with demand D decreases $H(t)$ with $\min\{D, (\lfloor \frac{L}{T} \rfloor + 1)Q_{max}\}$ (combining the first effect on $I(t)$ with the first effect on $K(t)$).
- A customer who arrived at epoch $t + L - nT\,(n = \lfloor \frac{L}{T} \rfloor + 1, \lfloor \frac{L}{T} \rfloor + 2, \ldots)$ with demand $D > nQ_{max}$ decreases $H(t)$ with $\min\{D - nQ_{max}, Q_{max}\}$ (third effect on $K(t)$).

Now we conclude that the amounts by which $H(t)$ decreases (the 'deliveries') are generated by either new customers or by old customers. To approximate the 'delivery' process we again take the viewpoint of section 2, where the delivery process is considered as being a superposition of offspring processes. However, note that the jump sizes are different in this situation. Denoting an arbitrary jump size of $H(t)$ by H and following the same line of reasoning as in section 2 and using (3.10) for the first moment of H we conclude that (compare formulas (2.2),(2.5) and (2.6))

$$\begin{aligned} \mathbb{E}H &= \frac{\lambda}{\bar{\lambda}} \mathbb{E}D_{0,\lfloor \frac{L}{T} \rfloor + 1} + \sum_{i=\lfloor \frac{L}{T} \rfloor + 1}^{\infty} \frac{\lambda}{\bar{\lambda}} \mathbb{E}D_{i,1} \\ &= \frac{\lambda}{\bar{\lambda}} \Big(\sum_{j=0}^{\lfloor \frac{L}{T} \rfloor} \mathbb{E}D_{j,1} + \sum_{j=\lfloor \frac{L}{T} \rfloor + 1}^{\infty} \mathbb{E}D_{j,1} \Big) = \frac{\lambda}{\bar{\lambda}} \mathbb{E}D \end{aligned} \tag{4.3}$$

$$\mathbb{E}H^n = \frac{\lambda}{\bar{\lambda}} \Big(\mathbb{E}\bar{D}^n_{0,\lfloor \frac{L}{T} \rfloor + 1} + \sum_{i=\lfloor \frac{L}{T} \rfloor + 1}^{\infty} \mathbb{E}\bar{D}^n_{i,1} \Big) \qquad (n = 2, 3, \ldots) \tag{4.4}$$

where $\tilde{\lambda} = \lambda + \sum_{i=\lfloor \frac{L}{T} \rfloor + 1}^{\infty} \lambda_i$ and λ_i is defined by (2.4). Analogous to (2.9) and (2.10) we conclude

$$\mathbb{E}U \simeq \frac{\mathbb{E}H^2}{2\mathbb{E}H} \tag{4.5}$$

$$\mathbb{E}U^2 \simeq \frac{\mathbb{E}H^3}{3\mathbb{E}H} \tag{4.6}$$

Together the formulas (4.1),(4.2),(4.5) and (4.6) enable us to apply the reorder point calculation method according to 1.3.

5. Validation of the approximations

To validate the quality of the approximations described in sections 2 and 3 we compare the results with discrete event simulation. First we computed s by the methods described in sections 2 and 3. Then with simulation the actual service levels were determined as well as the average stock on hand for the reorder points generated by the approximative methods. For each experiment we simulated 25 times 100.000 customers to guarantee a confidence interval of maximal 1 % of the actual service level. In this section we considered 240 cases as follows. The average customer demand size is fixed at 50, the coefficient of variation of the customer demand size (c_D) varies among 1,2 and 4 to emphasize the high variability in the demand size. The average number of customer orders (λ) has been taken as 1 per day. The lead time of replenishment orders (L) varies between 10 and 20 days. The replenishment order quantity is taken equal to 1000 for all cases. We adjust the P_2-service level (β) as 0.90 and 0.99. The intershipment time T is varied as 0.3, 0.5, 1 and 1.5 times the lead time. The maximum lotsize of a shipment (Q_{max}) is varied as 0.5, 1, 2 and 4 times the average customer demand. (see Appendix 9 and 10 for specification).
For the particular case $L = 20$, $\beta = 0.99$ and $c_D = 2$ the results are illustrated in Figure 5.1.

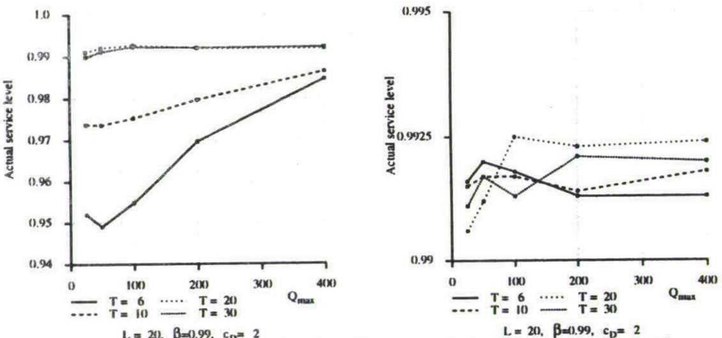

Figure 5.1. Actual service levels with the 'quick and dirty' method — *Actual service levels with the more advanced method*

The illustrations above, as well as results from Appendix 9, show the poor quality of the 'quick and dirty' method for $T < L$. For $T \geq L$, however, the 'quick and dirty' method is satisfactory. It can be shown that for $T \geq L$ both methods give the same results (see Remark 3.1). This, however, only holds in case a replenishment orders is handled before a delivery in case they coincide in time (see Remark 3.2). We notice the good performance of the advanced method for all values of T. The good performance is confirmed by Figure 5.2, where β is varied for some values of Q_{max}.

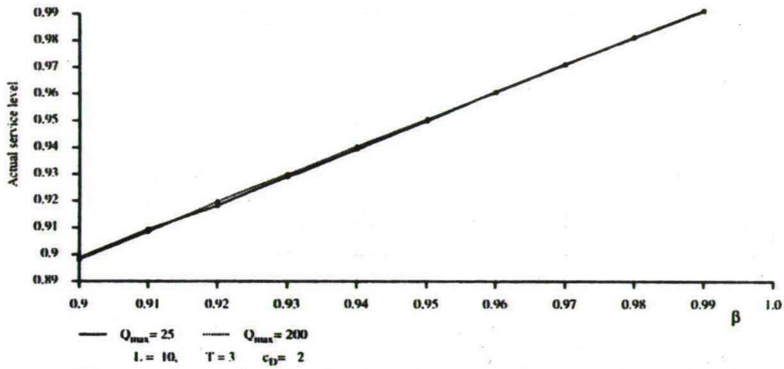

Figure 5.2. Actual service levels generated with the more advanced method.

To give an impression about the advantages of delivery splitting, we consider the stock reductions obtained. On the other hand we note that the disadvantages of delivery splitting is the increasing frequency of deliveries and the corresponding increase of transportation costs. Delivery splitting is particular effective in a situation where the demand patterns are affected by the negotiation process described in the introduction. De Kok analyzed the sales patterns of about 10.000 consumer electronic products in 13 European countries, and found that fast moving products showed rather erratic demand patterns. When delivery splitting is applied, for fast moving products, the shipment frequency remains the same because full truck loads of one product are changed in full truck loads of a number of products for the same customer. Furthermore co-ordinated supply of a number of customers may increase the shipment frequency for each product separately but does not increase the overall shipment frequency. Handling costs, on the other hand, are only slightly affected when shipments consists of complete pallets only (or standard package sizes). Thus, only to a limited extent cases there are cases where implementing delivery splitting will increase the transportation and handling costs, implying that cost reductions due to stock reductions can completely be considered as a profit due to delivery splitting. However, in case the transportation costs are relevant, the increase in the delivery intensity can be calculated easily as it equals $\lambda^* - \lambda$ (see section 2).

In Figure 5.3 we show the relative stock reductions due to delivery splitting, expressed in the percentage of the stock on hand that is needed when no delivery splitting is applied. To be more precise the figure represents the quantity $(\mathbb{E}I_\infty - \mathbb{E}I_{ni})/\mathbb{E}I_\infty \times 100\%$ where I_∞ denotes the average stock on hand level without delivery splitting and I_{ni} denotes the average stock on hand level with delivery splitting without use of future information.
We used the more advanced method to calculate the reorder point and used simulation to compute the average stock on hand, which is compared with the results of no delivery splitting.

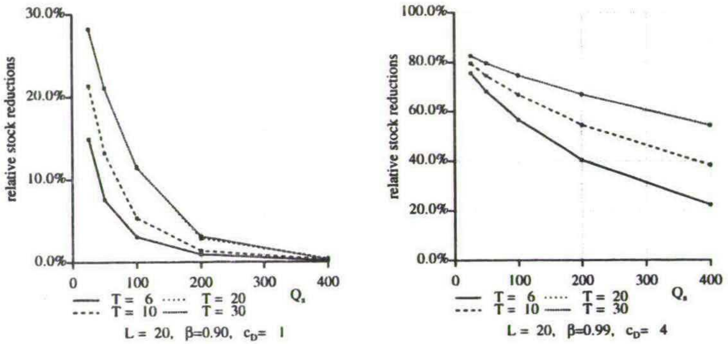

Figure 5.3. Stock reductions obtained by delivery splitting.

We conclude that in case c_D is relatively small delivery splitting does not reduce the average stock on hand significantly. However, for c_D large, which is particular the case in the situations described in the introduction, we notice the enormous stock on hand reductions that can be obtained. For an extensive treatment of the profitability (stock reductions weighted against increasing delivery frequency) we refer to De Kok en Janssen (1995).

To verify the quality of the method described in section 4 we again compare the results with simulation. We used the same approach as above. We considered again the 240 cases described above (for detailed results see Appendix 11).
Some condensed results are illustrated below in Figure 5.4.

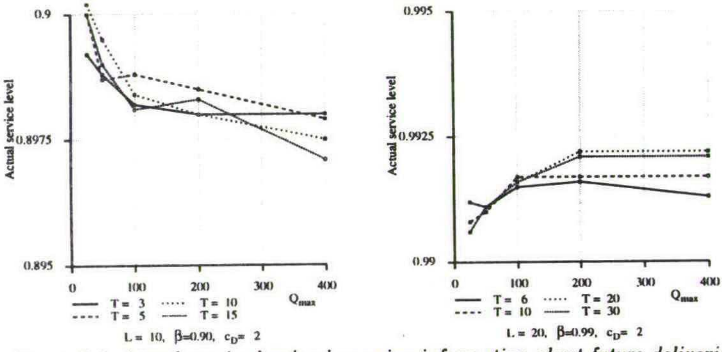

Figure 5.4. Actual service levels when using information about future deliveries with delivery splitting

As for the case in which no information about future deliveries is used, the advanced method also performs very good in case where information about future deliveries is used.

To compare the effectiveness of using information about future deliveries explicitly, we adjust the advanced method from section 3 for the case a delivery is handled before a replenishment order in case they coincide in time (see Remark 3.2). In Figure 5.5 we present the relative stock reductions obtained by delivery splitting in case we use information about future deliveries over the stock reductions obtained by delivery splitting without using information about future deliveries. To be more precise the figure represents the quantity $(IIEI_{ni} - IIEI_i)/(IIEI_{\infty} - IIEI_{ni}) \times 100\%$ where I_i denotes the average stock on hand level with delivery splitting using information about future deliveries explicitely.

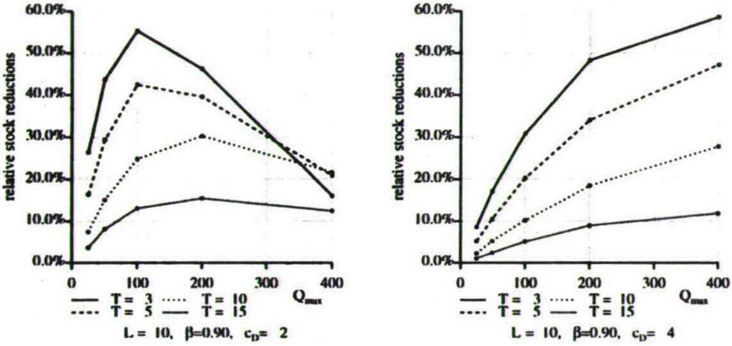

Figure 5.5. Additional stock reductions obtained by using information about future deliveries

It is clear that the additional stock reductions are dependent of Q_{max}. Actually, we conjecture that there exists a Q_{max} for which the additional stock reductions are maximal. The additional stock reduction increase for small Q_{max} because of the increasing amount known to be delivered during the lead time (the number of deliveries within the lead time remains the same but the quantity per delivery increases). Thus the information about future deliveries is used more effectively. On the other hand, for large Q_{max} the total amount known to be delivered during the lead time decreases, because the number of future deliveries decreases.

In deciding whether to implement delivery splitting with or without using the information about future deliveries a trade-off has to be made between the additional stock on hand savings and the extra cost due to a more complex replenishment strategy.

6. Conclusions and future research

In this paper we considered (s, Q) replenishment policies for an inventory control model with delivery splitting. Large demands are not delivered in one single batch but in a number of equally spaced deliveries at distance T from each other and each of size Q_{max} (except possibly the last). Two approximative methods are proposed to calculate for given value of the reorder quantity Q the minimal level of the reorder point s that guarantees a prespecified P_2 service level constraint. The 'quick and dirty' method that is proposed and analysed in section 2 only performs satisfactorily when $T \geq L$. The second more sophisticated method proposed in section 3 has an excellent performance irrespective the relation between T and L. The delivery splitting process provides explicit knowledge about future deliveries. This knowledge can be exploited in setting the reorder parameters. A variant of the method of section 3 is developed to calculate the appropriate reorder level for the situation in which the information about future deliveries is explicitly used in the replenishment process. Also this method shows excellent performance over a large range of parameter values. Through simulation it is shown that delivery splitting leads to substantial reduction in the stock level, in particular when the coefficient of variation is larger than 2. The resulting gains should be weighted against the increase in delivery costs (due to more frequent deliveries). Based on practical experience the authors know that quite often highly irregular demand patterns occur, which are extremely suitable for delivery splitting. However, the practical use of delivery splitting depends on the availability of an efficient method to calculate the appropriate reorder points. The methods proposed in sections 2 and 3 satisfy this need.

Several extensions are worthwhile to be considered. The generalisation to stochastic leadtimes (with the non-overtaking restriction) for the methods described in section 2 and 3 are straightforward. For the replenishment strategy based on the knowledge of future deliveries a stochastic leadtime implies that the total demand during the leadtime due to previously splitted orders becomes a stochastic variable, which seems to change the character of the replenishment strategy. Future research will be devoted to this question. Another important topic for future research is the development of an efficient (approximative) computational scheme to calculate the average expected stock level under delivery splitting. Due to the fact that there are at least four control variables (s, Q, T, and Q_{max}) it would be extremely helpful to have an efficient method to evaluate various scenarios. (Note that in our presentation the average stock levels had to be calculated through simulation. Finally we note that by delivery splitting not only the stock level of the manufacturer is decreased but also those of the customers. This observation calls for an analysis of the effects of delivery splitting in a multi-echelon context.

References

COX, D.R. 1962. *Renewal theory.*, Methuen, London.

DE KOK, A.G. 1991. Basics of inventory management: part 3 The (b,Q)-model. Research memorandum, Center for Economic Research, Tilburg University.

DE KOK, A.G. AND JANSSEN, F.B.S.L.P. 1995. Demand management in a multi-stage distribution chain. Research report, Graduate School of Industrial Engineering and Management Science, Eindhoven University of Technology.

HONG, J.D. AND HAYYA, J.C. 1992. Just-in-time purchasing: Single or multiple sourcing?'. *Int. J. of Prod. Econ.* **27**, 175-181.

KELLE, P. AND SILVER, E.A. 1990. Decreasing expected shortages through order splitting. *Eng. Costs and Production Economics* **19**, 351-357.

LAU, H.S. AND ZHAO, L.G. 1993. Optimal ordering policies with two suppliers when lead times and demand are all stochastic. *Eur. J. Opnl. Res.* **68**, 120-133.

RAMASESH ET. AL. 1991. Sole versus dual sourcing in stochastic lead-time (s, Q) inventory models. *Mgmt. Sci.* **37**, 428-443.

SCULLI, D. AND WU, S.Y. 1981. Stock control with two suppliers and normal lead times. *J. Opns. Res. Soc.* **32**, 1003-1009.

SILVER, E.A. AND PETERSON, R. 1985. *Decision Systems for Inventory Management and Production Planning.* Wiley, New York.

TIJMS, H.C. AND GROENEVELT, H. 1984. Simple approximations for the reorder point in periodic and continuous review (s, S) inventory systems with service level constraints', *Eur. J. Opnl. Res.* **17**, 175-190.

TIJMS, H.C. 1994. *Stochastic Models: An Algorithmic Approach.* Wiley, Chichester.

Appendix 1. (Two-moments fitting of a mixture of two Erlang distributions with the same scale parameter)

Let X be a positive random variable. Suppose the first two moments are known, i.e. $\mathbb{E}(X)$ and c_X^2. In this appendix the two-moment fit is given on the $E_{l,k}$ distribution
The density function of $E_{l,k}$ has the following form:

$$f(t) = p\mu^l \frac{t^{l-1}}{(l-1)!} e^{-\mu t} + (1-p)\mu^k \frac{t^{k-1}}{(k-1)!} e^{-\mu t} \quad (t \geq 0\,, 0 \leq p \leq 1)$$

If the coefficient of variation c_X is less than one l is set to $k-1$ else l is set to one. First consider $c_X < 1$, choosing the parameters k,p and μ successively as

$$\begin{aligned} k &= \min\{\, m \,|\, \frac{1}{m} \leq c_X^2 \leq \frac{1}{m-1} \,\} \\ p &= \frac{kc_X^2 - \sqrt{k(1+c_X^2) - k^2 c_X^2}}{1+c_X^2} \\ \mu &= \frac{k-p}{EX} \end{aligned}$$

The associated $E_{k-1,k}$ distribution fits the first two moments of X. The $E_{k-1,k}$ distribution is always unimodal, only coefficients of variations between 0 and 1 can be achieved and for $k > 2$ the density has the same shape as the gamma distribution When $c_X \geq 1$ choosing the parameters k,p and μ successively as

$$\begin{aligned} k &= \min\{\, m \,|\, m \geq 2\,, c_X^2 \leq \frac{m^2+4}{4m} \,\} \\ p &= \frac{2lc_X^2 + l - 2 - \sqrt{l^2 + 4 - 4lc_X^2}}{2(l-1)(1+c_X^2} \\ \mu &= \frac{p + l(1-p)}{EX} \end{aligned}$$

The $E_{1,k}$ distribution with the parameters set as above is almost always bimodel this in contradiction to the hyperexponential distribution.

Appendix 2. (Proof of relations (2.1) and (2.3))

$$
\begin{aligned}
\mathbb{E}Z^*(t) &= \mathbb{E}\Big(\sum_{j=1}^{N^*(t)} D_j^*\Big) \\
&= \mathbb{E}(N^*(t))\,\mathbb{E}D_1^* \\
&= \lambda^* t\,\mathbb{E}D_1^*
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}Z_i(t) &= \mathbb{E}\Big(\sum_{j=1}^{N_i(t)} D_{i,j}\Big) \\
&= \mathbb{E}(N_i(t))\,\mathbb{E}D_{i,1} \\
&= \lambda_i t\,\mathbb{E}D_{i,1}
\end{aligned}
$$

Taking the expectations at both sides of the equation (2.1) it follows that

$$
\mathbb{E}Z^*(t) = \lambda^* t\,\mathbb{E}D_1^* = \sum_{i=0}^{\infty} \lambda_i t\,\mathbb{E}D_{i,1}
$$

Thus

$$
\mathbb{E}D_1^* = \sum_{i=0}^{\infty} \frac{\lambda}{\lambda^*}\mathbb{E}D_{i,1}
$$

and analogously

$$
\mathbb{E}D_1^{*n} = \sum_{i=0}^{\infty} \frac{\lambda_i}{\lambda^*}\mathbb{E}D_{i,1}^n \quad n = 2, 3, \ldots
$$

Appendix 3. (Proof of relations (2.5) and (2.6))

Expressions for $I\!E D_{i,1}$ and $I\!E D^2_{i,1}$ are obtained as follows:

$$\begin{aligned}
& I\!E D_{i,1} \\
= \; & I\!E((D - iQ_{max})I(iQ_{max} \le D < (i+1)Q_{max}) + Q_{max}I(D \ge (i+1)Q_{max}) | D > iQ_{max}) \\
= \; & \frac{\int_{iQ_{max}}^{(i+1)Q_{max}} (x - iQ_{max}) dF_D(x) + \int_{(i+1)Q_{max}}^{\infty} Q_{max} dF_D(x)}{I\!P(D > iQ_{max})} \\
= \; & \frac{\int_{iQ_{max}}^{(i+1)Q_{max}} x dF_D(x) - iQ_{max} \int_{iQ_{max}}^{(i+1)Q_{max}} dF_D(x) + Q_{max}(1 - F_D((i+1)Q_{max})}{1 - F_D(iQ_{max})} \\
= \; & \frac{\int_{iQ_{max}}^{(i+1)Q_{max}} x dF_D(x) + iQ_{max}\Big(\big(1 - F_D((i+1)Q_{max})\big) - \big(1 - F_D(iQ_{max})\big)\Big) + Q_{max}(1 - F_D((i+1)Q_{max}))}{1 - F_D(iQ_{max})} \\
= \; & \frac{\int_{iQ_{max}}^{(i+1)Q_{max}} x dF_D(x) + Y_{i+1} - Y_i}{1 - F_D(iQ_{max})}
\end{aligned}$$

with $Y_i = iQ_{max}(1 - F_D(iQ_{max}))$
analogously

$$\begin{aligned}
& I\!E D^2_{i,1} \\
= \; & I\!E((D - iQ_{max})^2 I\{iQ_{max} \le D < (i+1)Q_{max}\} + Q^2_{max} I\{D \ge (i+1)Q_{max}\} | D \ge iQ_{max}) \\
= \; & \frac{\int_{iQ_{max}}^{(i+1)Q_{max}} (x - iQ_{max})^2 dF_D(x) + \int_{(i+1)Q_{max}}^{\infty} Q^2_{max} dF_D(x)}{1 - F_D(iQ_{max})} \\
= \; & \frac{W_i - 2iQ^2_{max}(1 - F_D(iQ_{max})) + (iQ_{max})^2(1 - F_D(iQ_{max})) - (i^2 - 1)Q^2_{max}(1 - F_D((i+1)Q_{max}))}{1 - F_D(iQ_{max})} \\
= \; & \frac{W_i + i(i-2)Q^2_{max}\big(1 - F_D(iQ_{max})\big) - (i+1)(i+1-2)Q^2_{max}\big(1 - F_D((i+1)Q_{max})\big)}{1 - F_D(iQ_{max})} \\
= \; & \frac{W_i + Z_i - Z_{i+1}}{1 - F_D(iQ_{max})}
\end{aligned}$$

with $W_i = \int_{iQ_{max}}^{(i+1)Q_{max}} (x^2 - 2iQ_{max}x) dF_D(x) + 2iQ^2_{max}(1 - F_D(iQ_{max}))$
and $Z_i = (i(i-2)Q^2_{max})(1 - F_D(iQ_{max}))$
Subsitution of the $I\!E D_{i,1}$ and $I\!E D^2_{i,1}$ for $i = 0, 1, \ldots$ in the expressions for $I\!E D^*_1$ and $I\!E D^{*2}_1$ and using (2.4) yields

$$\begin{aligned}
I\!E D^*_1 &= \sum_{i=0}^{\infty} \frac{\lambda_i}{\lambda^*} I\!E D_{i,1} \\
&= \sum_{i=0}^{\infty} \frac{\lambda}{\lambda^*} \Big(\int_{iQ_{max}}^{(i+1)Q_{max}} x dF_D(x) - Y_i + Y_{i+1} \Big)
\end{aligned}$$

$$= \frac{\lambda}{\lambda^*}\mathbb{E}D$$

and

$$\begin{aligned}
\mathbb{E}D_1^{*2} &= \sum_{i=0}^{\infty}\frac{\lambda_i}{\lambda^*}\mathbb{E}D_{i,1}^2 \\
&= \frac{\lambda}{\lambda^*}\sum_{i=0}^{\infty}\Big(\int_{iQ_{max}}^{(i+1)Q_{max}} x^2 dF_D(x) - 2Q_{max}(i\int_{iQ_{max}}^{(i+1)Q_{max}} x dF_D(x) - \int_{iQ_{max}}^{\infty} iQ_{max} dF_D(x)) + Z_i - Z_{i+1}\Big) \\
&= \frac{\lambda}{\lambda^*}\Big(\mathbb{E}D^2 - 2Q_{max}\sum_{i=1}^{\infty}\int_{iQ_{max}}^{\infty}(x - iQ_{max})dF_D(x)\Big) \\
&= \frac{\lambda}{\lambda^*}\Big(\mathbb{E}D^2 - 2Q_{max}\sum_{i=1}^{\infty}\int_{iQ_{max}}^{\infty}(1 - F_D(x))dx\Big)
\end{aligned}$$

Appendix 4. (Expressions for the first three moments of D^* in case of a generalized Erlang distribution)

Using (1.1) for the specification of F_D we find the following explicit expressions for λ^* and $\mathbb{E}D_1^{*2}$:

$$\lambda^* = \lambda\sum_{i=0}^{\infty}\Big(pI(k, iQ_{max}, \mu) + (1-p)I(l, iQ_{max}, \mu)\Big) \tag{A.4.1}$$

$$\begin{aligned}\mathbb{E}D_1^{*2} &= \frac{\lambda}{\lambda^*}(\mathbb{E}D^2 - 2pQ_{max}\sum_{i=1}^{\infty}(\frac{k}{\mu}I(k+1, iQ_{max}, \mu) - iQ_{max}I(k, iQ_{max}, \mu) \\ &\quad -2(1-p)Q_{max}\sum_{i=1}^{\infty}(\frac{l}{\mu}I(l+1, iQ_{max}, \mu) - iQ_{max}I(l, iQ_{max}, \mu))\end{aligned} \tag{A.4.2}$$

where $I(k, z, \mu) := \sum_{j=0}^{k-1}\frac{(\mu z)^j}{j!}e^{-\mu z}$.

Along the same lines it follows straightforwardly, that

$$\begin{aligned}\mathbb{E}D_1^{*3} &= \frac{\lambda}{\lambda^*}\Bigg(\mathbb{E}D^3 - \sum_{i=0}^{\infty}3iQ_{max}\Big(p\frac{(k+1)k}{\mu_1^2}J(k+2, i, \mu_1) + (1-p)\frac{(l+1)l}{\mu_2^2}J(l+2, i, \mu_2)\Big) \\ &\quad + \sum_{i=0}^{\infty}3(iQ_{max})^2\Big(p\frac{k}{\mu_1}J(k+1, i, \mu_1) + (1-p)\frac{l}{\mu_2}J(l+1, i, \mu_2)\Big) \\ &\quad - \sum_{i=0}^{\infty}(iQ_{max})^3\Big(pJ(k, i, \mu_1) + (1-p)J(l, i, \mu_2)) \\ &\quad + (Q_{max})^3\sum_{i=0}^{\infty}\Big(pI(k, (i+1)Q_{max}, \mu_1) + (1-p)I(l, (i+1)Q_{max}, \mu_2)\Big)\Bigg)\end{aligned} \tag{A.4.3}$$

where $J(k, i, \mu) := I(k, iQ_{max}, \mu) - I(k, (i+1)Q_{max}, \mu)$

Appendix 5. Proof of relation (3.10).

We first proof that the following relation holds

$$\bar{D}_{k,l} = \bar{D}_{k,1} + \bar{D}_{k+1,l-1} \qquad k = 0, 1, ..., l = 1, 2, ... \tag{A.5.1}$$

For $l = 1$ it easily follows that $\bar{D}_{k,1} = \bar{D}_{k,1} + \bar{D}_{k+1,0} = \bar{D}_{k,1}$.
For $l > 1$ we distinguish four cases

$$\begin{array}{lll}
D < kQ_{max} & : & \bar{D}_{k,l} = 0 \\
 & : & \bar{D}_{k,1} = \bar{D}_{k+1,l-1} = 0 \\
kQ_{max} \leq D < (k+1)Q_{max} & : & \bar{D}_{k,l} = D - kQ_{max} \\
 & : & \bar{D}_{k,1} = D - kQ_{max} \\
 & : & \bar{D}_{k+1,l-1} = 0 \\
(k+1)Q_{max} \leq D < lQ_{max} & : & \bar{D}_{k,l} = D - kQ_{max} \\
 & : & \bar{D}_{k,1} = Q_{max} \\
 & : & \bar{D}_{k+1,l-1} = D - (k+1)Q_{max} \\
D \geq lQ_{max} & : & \bar{D}_{k,l} = lQ_{max} \\
 & : & \bar{D}_{k,1} = Q_{max} \\
 & : & \bar{D}_{k+1,l-1} = (l-1)Q_{max}
\end{array}$$

□

Using (A.5.1) l times yield to

$$\bar{D}_{k,l} \;=\; \sum_{j=k}^{k+l-1} \bar{D}_{j,1} \quad k, l = 0, 1, 2, ... \tag{A.5.2}$$

Appendix 6. (Proof of formulas 3.11 and 3.12)

Let $N := (N_i^{(1)}, i = 1, 2, ..., \lfloor \frac{L}{T} \rfloor + 1 \,;\, N_{i,A}^{(2)}, i = 1, 2, ...;\, N_{i,B}^{(2)}, i = 1, 2, ...)$ Then

$$\mathbb{E}(Z(L)|N) = \mathbb{E}\left(\sum_{i=1}^{\lfloor \frac{L}{T} \rfloor + 1} \sum_{j=1}^{N_i^{(1)}} D_{i,j}^{(1)} + \sum_{i=1}^{\infty} \left(\sum_{j=1}^{N_{i,A}^{(2)}} D_{i,j,A}^{(2)} + \sum_{j=1}^{N_{i,B}^{(2)}} D_{i,j,B}^{(2)} \right) \Big| N \right)$$

Using (3.6) to (3.8) we conclude

$$\mathbb{E}(Z(L)|N) = \sum_{i=1}^{\lfloor \frac{L}{T} \rfloor + 1} N_i^{(1)} \mathbb{E}\bar{D}_{0,i} + \sum_{i=1}^{\infty} \left(N_{i,A}^{(2)} \mathbb{E}\bar{D}_{i,\lfloor \frac{L}{T} \rfloor} + N_{i,B}^{(2)} \mathbb{E}\bar{D}_{i,\lfloor \frac{L}{T} \rfloor + 1} \right)$$

$$\begin{aligned}
\mathbb{E}Z(L) &= \mathbb{E}(\mathbb{E}(Z(L)|N)) \\
&= \mathbb{E}\left(\sum_{i=1}^{\lfloor \frac{L}{T} \rfloor + 1} N_i^{(1)} \mathbb{E}\bar{D}_{0,i} + \sum_{i=1}^{\infty} \left(N_{i,A}^{(2)} \mathbb{E}\bar{D}_{i,\lfloor \frac{L}{T} \rfloor} + N_{i,B}^{(2)} \mathbb{E}\bar{D}_{i,\lfloor \frac{L}{T} \rfloor + 1} \right) \right) \\
&= \sum_{i=1}^{\lfloor \frac{L}{T} \rfloor + 1} \mathbb{E}N_i^{(1)} \mathbb{E}\bar{D}_{0,i} + \sum_{i=1}^{\infty} \left(\mathbb{E}N_{i,A}^{(2)} \mathbb{E}\bar{D}_{i,\lfloor \frac{L}{T} \rfloor} + \mathbb{E}N_{i,B}^{(2)} \mathbb{E}\bar{D}_{i,\lfloor \frac{L}{T} \rfloor + 1} \right)
\end{aligned}$$

Since the customer arrival process is a Poisson process it follows that

$$\begin{aligned}
\mathbb{E}N_i^{(1)} &= \sigma^2(N_i^{(1)}) &&= \lambda T && i = 1, 2, ..., \lfloor \tfrac{L}{T} \rfloor \\
\mathbb{E}N_{\lfloor \frac{L}{T} \rfloor + 1}^{(1)} &= \sigma^2(N_{\lfloor \frac{L}{T} \rfloor + 1}^{(1)}) &&= \lambda \xi && \\
\mathbb{E}N_{i,A}^{(2)} &= \sigma^2(N_{i,A}^{(2)}) &&= \lambda(T - \xi) && i = 1, 2, ... \\
\mathbb{E}N_{i,B}^{(2)} &= \sigma^2(N_{i,B}^{(2)}) &&= \lambda \xi && i = 1, 2, ...
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}Z(L) &= \sum_{i=1}^{\lfloor \frac{L}{T} \rfloor} \lambda T \mathbb{E}\bar{D}_{0,i} + \lambda \xi \mathbb{E}\bar{D}_{0,\lfloor \frac{L}{T} \rfloor + 1} \\
&\quad + \sum_{i=1}^{\infty} \left(\lambda(T - \xi) \mathbb{E}\bar{D}_{i,\lfloor \frac{L}{T} \rfloor} + \lambda \xi \mathbb{E}\bar{D}_{i,\lfloor \frac{L}{T} \rfloor + 1} \right) \qquad \text{(A.6.1)}
\end{aligned}$$

Using (3.10), where we take at both sides the expectation, we can rewrite (A.6.1) as follows

$$\begin{aligned}
\mathbb{E}Z(L) &= \sum_{i=1}^{\lfloor \frac{L}{T} \rfloor} \lambda T \mathbb{E}\bar{D}_{0,i} + \lambda \xi \mathbb{E}\bar{D}_{0,\lfloor \frac{L}{T} \rfloor + 1} + \sum_{i=1}^{\infty} \left(\lambda(T - \xi) \mathbb{E}\bar{D}_{i,\lfloor \frac{L}{T} \rfloor} + \lambda \xi \mathbb{E}\bar{D}_{i,\lfloor \frac{L}{T} \rfloor + 1} \right) \\
&= \sum_{i=1}^{\lfloor \frac{L}{T} \rfloor} \sum_{j=0}^{i-1} \lambda T \mathbb{E}\bar{D}_{j,1} + \sum_{j=0}^{\lfloor \frac{L}{T} \rfloor} \lambda \xi \mathbb{E}\bar{D}_{j,1} \\
&\quad + \sum_{i=1}^{\infty} \left(\lambda(T - \xi) \mathbb{E} \sum_{j=i}^{i + \lfloor \frac{L}{T} \rfloor - 1} \bar{D}_{j,1} + \lambda \xi \sum_{j=i}^{i + \lfloor \frac{L}{T} \rfloor} \mathbb{E}\bar{D}_{j,1} \right) \\
&= \sum_{j=0}^{\lfloor \frac{L}{T} \rfloor - 1} \sum_{i=j+1}^{\lfloor \frac{L}{T} \rfloor} \lambda T \mathbb{E}\bar{D}_{j,1} + \sum_{j=0}^{\lfloor \frac{L}{T} \rfloor} \lambda \xi \mathbb{E}\bar{D}_{j,1}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{\infty}\sum_{j=i}^{i+\lfloor\frac{L}{T}\rfloor-1}\lambda T\mathbb{E}\bar{D}_{j,1}+\sum_{i=1}^{\infty}\lambda\xi\mathbb{E}\bar{D}_{i+\lfloor\frac{L}{T}\rfloor,1}\\
=&\sum_{j=0}^{\lfloor\frac{L}{T}\rfloor-1}\sum_{i=j+1}^{\lfloor\frac{L}{T}\rfloor}\lambda T\mathbb{E}\bar{D}_{j,1}+\sum_{j=0}^{\infty}\lambda\xi\mathbb{E}\bar{D}_{j,1}\\
&+\sum_{j=0}^{\lfloor\frac{L}{T}\rfloor-1}\sum_{i=1}^{j}\lambda T\mathbb{E}\bar{D}_{j,1}+\sum_{j=\lfloor\frac{L}{T}\rfloor}^{\infty}\sum_{i=j-\lfloor\frac{L}{T}\rfloor+1}^{j}\lambda T\mathbb{E}\bar{D}_{j,1}\\
=&\sum_{j=0}^{\infty}\lambda(\lfloor\frac{L}{T}\rfloor T+\xi)\bar{D}_{j,1}\\
=&\lambda L\mathbb{E}D
\end{aligned}
$$

For $\sigma^2(Z(L))$ we derive

$$
\sigma^2(Z(L)|N=n) = \sigma^2\Bigg(\sum_{i=1}^{\lfloor\frac{L}{T}\rfloor+1}\sum_{j=1}^{N_i^{(1)}}D_{i,j}^{(1)}+\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{N_{i,A}^{(2)}}D_{i,j,A}^{(2)}+\sum_{j=1}^{N_{i,B}^{(2)}}D_{i,j,B}^{(2)}\Big)\Bigg|N\Bigg)
$$

Hence it follows from (3.6) to (3.8) that

$$
\sigma^2(Z(L)|N) = \sum_{i=1}^{\lfloor\frac{L}{T}\rfloor+1}N_i^{(1)}\sigma^2(\bar{D}_{0,i})+\sum_{i=1}^{\infty}\Big(N_{i,A}^{(2)}\sigma^2(\bar{D}_{i,\lfloor\frac{L}{T}\rfloor})+N_{i,B}^{(2)}\sigma^2(\bar{D}_{i,\lfloor\frac{L}{T}\rfloor+1})\Big)
$$

which implies that

$$
\begin{aligned}
\sigma^2(Z(L)) =& \mathbb{E}(\sigma^2(D_L^*|N))+\sigma^2(\mathbb{E}(D_L^*|N))\\
=& \mathbb{E}\Bigg(\Bigg(\sum_{i=1}^{\lfloor\frac{L}{T}\rfloor+1}N_i^{(1)}\sigma^2(\bar{D}_{0,i})+\sum_{i=1}^{\infty}\Big(N_{i,A}^{(2)}\sigma^2(\bar{D}_{i,\lfloor\frac{L}{T}\rfloor})+N_{i,B}^{(2)}\sigma^2(\bar{D}_{i,\lfloor\frac{L}{T}\rfloor+1})\Big)\Bigg)\\
&+\sigma^2\Bigg(\sum_{i=1}^{\lfloor\frac{L}{T}\rfloor+1}N_i^{(1)}\mathbb{E}\bar{D}_{0,i}+\sum_{i=1}^{\infty}\Big(N_{i,A}^{(2)}\mathbb{E}\bar{D}_{i,\lfloor\frac{L}{T}\rfloor}+N_{i,B}^{(2)}\mathbb{E}\bar{D}_{i,\lfloor\frac{L}{T}\rfloor+1}\Big)\Bigg)\\
=& \sum_{i=1}^{\lfloor\frac{L}{T}\rfloor+1}\mathbb{E}N_i^{(1)}\sigma^2(\bar{D}_{0,i})+\sum_{i=1}^{\infty}\Big(\mathbb{E}N_{i,A}^{(2)}\sigma^2(\bar{D}_{i,\lfloor\frac{L}{T}\rfloor})+\mathbb{E}N_{i,B}^{(2)}\sigma^2(\bar{D}_{i,\lfloor\frac{L}{T}\rfloor+1})\Big)\\
&+\sum_{i=1}^{\lfloor\frac{L}{T}\rfloor+1}\sigma^2(N_i^{(1)})\mathbb{E}^2\bar{D}_{0,i}+\sum_{i=1}^{\infty}\Big(\sigma^2(N_{i,A}^{(2)})\mathbb{E}^2\bar{D}_{i,\lfloor\frac{L}{T}\rfloor}+\sigma^2(N_{i,B}^{(2)})\mathbb{E}^2\bar{D}_{i,\lfloor\frac{L}{T}\rfloor+1}\Big)\\
=& \sum_{i=1}^{\lfloor\frac{L}{T}\rfloor}\lambda T\mathbb{E}\bar{D}_{0,i}^2+\lambda\xi\mathbb{E}\bar{D}_{0,\lfloor\frac{L}{T}\rfloor+1}^2+\sum_{i=1}^{\infty}\Big(\lambda(T-\xi)\mathbb{E}\bar{D}_{i,\lfloor\frac{L}{T}\rfloor}^2+\lambda\xi\mathbb{E}\bar{D}_{i,\lfloor\frac{L}{T}\rfloor+1}^2\Big)
\end{aligned}
$$

Appendix 7. (Proof of asymptotic validity of (3.13))

Let $Y_i(s)$, $i = 0, 1, 2, \ldots$ be a compound Poisson process with Poisson parameter λ_i and demand size distribution $F_i(.)$ with mean $\mathbb{E}D_i$ and $F_i(0) = 0$. Assume that the processes $(Y_i(s))_{i=0}^{\infty}$ are independent and define $X(s) := \sum_{i=0}^{\infty} Y_i(s)$. We are interested in the probability
$q_i(t) :=$ probability that overshoot of $X(s)$ over the value t is caused by the i-th process.
Then $q_i(t)$ satisfies the following renewal type equation

$$q_i(t) = \frac{\lambda_i}{\lambda^*}(1 - F_i(t)) + \sum_{j=0}^{\infty} \frac{\lambda_j}{\lambda^*} \int_0^t q_i(t-x)dF_j(x) \tag{A.7.1}$$

Taking Laplace transforms on both sides yields:

$$\tilde{q}_i(s) = \frac{\lambda_i}{\lambda^* s}(1 - \tilde{F}_i(s)) + \sum_{j=0}^{\infty} \frac{\lambda_j \tilde{q}_i(s)}{\lambda^*} \tilde{F}_j(s) \tag{A.7.2}$$

where $\tilde{q}_i(s) := \int_0^{\infty} e^{-st} q_i(t)dt$ and $\tilde{F}_j(s) := \int_0^{\infty} e^{-st} dF_j(t)$
Solving (A.7.1) for $\tilde{q}_i(s)$ yields

$$\tilde{q}_i(s) = \frac{\frac{\lambda_i}{\lambda^*}(1 - \tilde{F}_i(s))}{1 - \sum_{j=0}^{\infty} \frac{\lambda_j}{\lambda^*} \tilde{F}_j(s)} \tag{A.7.3}$$

Since

$$\lim_{t\to\infty} q_i(t) = \lim_{s\downarrow 0} s\tilde{q}_i(s) \tag{A.7.4}$$

we conclude from (A.7.2) that

$$\lim_{t\to\infty} q_i(t) = \lim_{s\downarrow 0} \frac{\frac{\lambda_i}{\lambda^*}(1 - \tilde{F}_i(s)}{1 - \sum_{j=0}^{\infty} \frac{\lambda_j}{\lambda^*} \tilde{F}_j(s)} \tag{A.7.5}$$

which implies, using l'Hopital's rule

$$\lim_{t\to\infty} q_i(t) = \frac{\lambda_i \mathbb{E}(D_i)}{\sum_{j=0}^{\infty} \lambda_j \mathbb{E}(D_j)}, \tag{A.7.6}$$

which proves the asymptotic validity of (3.13).

Appendix 8. (Proof of (3.17) and (3.18))

Using (3.13),(3.15) and (3.16) we find

$$
\begin{aligned}
\mathbb{E}U_i &= \mathbb{E}(\hat{D}_{i,\lceil \frac{L}{T}\rceil} - V_i) \\
&= \int_0^{\lceil \frac{L}{T}\rceil Q_{max}} \mathbb{E}(x - V_i) f_{\hat{D}_i}(x)dx + \int_{\lceil \frac{L}{T}\rceil Q_{max}}^{\infty} \mathbb{E}(\lceil \frac{L}{T}\rceil Q_{max} - V_i) f_{\hat{D}_i}(x)dx \\
&= \int_0^{Q_{max}} \int_0^{x} \frac{x-v}{x} dv f_{\hat{D}_i}(x)dx + \int_{Q_{max}}^{\lceil \frac{L}{T}\rceil Q_{max}} \int_0^{Q_{max}} \frac{x-v}{Q_{max}} dv f_{\hat{D}_i}(x)dx \\
&+ \int_{\lceil \frac{L}{T}\rceil Q_{max}}^{\infty} \int_0^{Q_{max}} \frac{\lceil \frac{L}{T}\rceil Q_{max} - v}{Q_{max}} dv f_{\hat{D}_i}(x)dx \\
&= \int_0^{Q_{max}} \frac{1}{2} x f_{\hat{D}_i}(x)dx + \int_{Q_{max}}^{\lceil \frac{L}{T}\rceil Q_{max}} \left((x - Q_{max}) + \frac{1}{2}Q_{max}\right) f_{\hat{D}_i}(x)dx \\
&+ \int_{\lceil \frac{L}{T}\rceil Q_{max}}^{\infty} (\lceil \frac{L}{T}\rceil - 1 + \frac{1}{2}) Q_{max} f_{\hat{D}_i}(x)dx \\
&= \mathbb{E}(\bar{D}_{i,1})^{-1}\Bigg(\int_0^{Q_{max}} \frac{1}{2}x^2 f_D(x + iQ_{max})dx + \frac{1}{2}Q_{max}^2 \int_{Q_{max}}^{\infty} f_D(x + iQ_{max})dx \\
&+ \int_{Q_{max}}^{\lceil \frac{L}{T}\rceil Q_{max}} Q_{max}(x - Q_{max}) f_D(x + iQ_{max})dx + \int_{\lceil \frac{L}{T}\rceil Q_{max}}^{\infty} (\lceil \frac{L}{T}\rceil - 1) Q_{max}^2 f_D(x + iQ_{max})dx \\
&= \mathbb{E}(D_{i,1})^{-1}\Bigg(\int_{iQ_{max}}^{(i+1)Q_{max}} \frac{1}{2}(x - iQ_{max})^2 f_D(x)dx + \frac{1}{2}Q_{max}^2 \int_{(i+1)Q_{max}}^{\infty} f_D(x)dx \\
&+ \int_{(i+1)Q_{max}}^{(\lceil \frac{L}{T}\rceil + i)Q_{max}} Q_{max}(x - (i+1)Q_{max}) f_D(x)dx + \int_{(\lceil \frac{L}{T}\rceil + i)Q_{max}}^{\infty} (\lceil \frac{L}{T}\rceil - 1) Q_{max}^2 f_D(x)dx \Bigg) \\
&= \frac{\frac{1}{2}\mathbb{E}D_{i,1}^2 + Q_{max}\mathbb{E}\bar{D}_{i+1,\lceil \frac{L}{T}\rceil - 1}}{\mathbb{E}\bar{D}_{i,1}}
\end{aligned}
$$

Analogously

$$
\begin{aligned}
\mathbb{E}U_i^2 &= \mathbb{E}((\hat{D}_{i,\lceil \frac{L}{T}\rceil} - V_i)^2) \\
&= \int_0^{\lceil \frac{L}{T}\rceil Q_{max}} \mathbb{E}((x - V_i)^2) f_{\hat{D}_i}(x)dx + \int_{\lceil \frac{L}{T}\rceil Q_{max}}^{\infty} \mathbb{E}((\lceil \frac{L}{T}\rceil Q_{max} - V_i)^2) f_{\hat{D}_i}(x)dx \\
&= \int_0^{Q_{max}} \int_0^{x} \frac{(x-v)^2}{x} dv f_{\hat{D}_i}(x)dx + \int_{Q_{max}}^{\lceil \frac{L}{T}\rceil Q_{max}} \int_0^{Q_{max}} \frac{(x-v)^2}{Q_{max}} dv f_{\hat{D}_i}(x)dx \\
&\quad + \int_{\lceil \frac{L}{T}\rceil Q_{max}}^{\infty} \int_0^{Q_{max}} \frac{(\lceil \frac{L}{T}\rceil Q_{max} - v)^2}{Q_{max}} dv f_{\hat{D}_i}(x)dx \\
&= \int_0^{Q_{max}} \frac{1}{3}x^2 f_{\hat{D}_i}(x)dx + \int_{Q_{max}}^{\lceil \frac{L}{T}\rceil Q_{max}} \left(x^2 - xQ_{max} + \frac{1}{3}Q_{max}^2\right) f_{\hat{D}_i}(x)dx \\
&\quad + \int_{\lceil \frac{L}{T}\rceil Q_{max}}^{\infty} ((\lceil \frac{L}{T}\rceil - 1)^2 + \lceil \frac{L}{T}\rceil - 1 + \frac{1}{3})Q_{max}^2 f_{\hat{D}_i}(x)dx \\
&= \mathbb{E}(\bar{D}_{i,1})^{-1}\Bigg(\int_0^{Q_{max}} \frac{1}{3}x^3 f_D(x + iQ_{max})dx + \frac{1}{3}Q_{max}^3 \int_{Q_{max}}^{\infty} f_D(x + iQ_{max})dx \\
&\quad + \int_{Q_{max}}^{\lceil \frac{L}{T}\rceil Q_{max}} Q_{max}x(x - Q_{max}) f_D(x + iQ_{max})dx \\
&\quad + \int_{\lceil \frac{L}{T}\rceil Q_{max}}^{\infty} ((\lceil \frac{L}{T}\rceil - 1)^2 + \lceil \frac{L}{T}\rceil - 1)Q_{max}^3 f_D(x + iQ_{max})dx \Bigg) \\
&= \mathbb{E}(\bar{D}_{i,1})^{-1}\Bigg(\frac{1}{3}\mathbb{E}(\bar{D}_{i,1}^3) + Q_{max}\Bigg(\int_{(i+1)Q_{max}}^{(\lceil \frac{L}{T}\rceil + i)Q_{max}} (x - (i+1)Q_{max})^2 f_D(x)dx \\
&\quad + \int_{(\lceil \frac{L}{T}\rceil + i)Q_{max}}^{\infty} ((\lceil \frac{L}{T}\rceil - 1)Q_{max})^2 f_D(x)dx \Bigg) \\
&\quad + Q_{max}^2 \Bigg(\int_{(i+1)Q_{max}}^{(\lceil \frac{L}{T}\rceil + i)Q_{max}} (x - (i+1)Q_{max}) f_D(x)dx + \int_{(\lceil \frac{L}{T}\rceil + i)Q_{max}}^{\infty} ((\lceil \frac{L}{T}\rceil - 1)Q_{max} f_D(x)dx) \Bigg)\Bigg) \\
&= \frac{\frac{1}{3}\mathbb{E}\bar{D}_{i,1}^3 + Q_{max}\mathbb{E}\bar{D}_{i+1,\lceil \frac{L}{T}\rceil - 1}^2 + Q_{max}^2 \mathbb{E}\bar{D}_{i+1,\lceil \frac{L}{T}\rceil - 1}}{\mathbb{E}D_{i,1}}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathbb{E}U &= \sum_{i=0}^{\infty} q_i \mathbb{E}U_i \\
&= \sum_{i=0}^{\infty} \frac{\mathbb{E}\bar{D}_{i,1}}{\mathbb{E}D}\left(\frac{\mathbb{E}\bar{D}_{i,1}^2}{2\mathbb{E}\bar{D}_{i,1}} + \frac{Q_{max}\mathbb{E}\bar{D}_{i+1,\lceil\frac{L}{T}\rceil-1}}{\mathbb{E}\bar{D}_{i,1}}\right) \\
&= \sum_{i=0}^{\infty}\left(\frac{\lambda_i \mathbb{E}D_{i,1}^2}{\lambda^* 2\mathbb{E}D^*} + \frac{Q_{max}\mathbb{E}\bar{D}_{i+1,\lceil\frac{L}{T}\rceil-1}}{\mathbb{E}D}\right) \\
&= \frac{\mathbb{E}D^{*2}}{2\mathbb{E}D^*} + \sum_{i=0}^{\infty}\frac{Q_{max}\mathbb{E}\bar{D}_{i+1,\lceil\frac{L}{T}\rceil-1}}{\mathbb{E}D}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}U^2 &= \sum_{i=0}^{\infty} q_i \mathbb{E}U_i^2 \\
&= \sum_{i=0}^{\infty} \frac{\mathbb{E}\bar{D}_{i,1}}{\mathbb{E}D}\left(\frac{\mathbb{E}\bar{D}_{i,1}^3}{\mathbb{E}3\bar{D}_{i,1}} + \frac{Q_{max}\mathbb{E}\bar{D}_{i+1,\lceil\frac{L}{T}\rceil-1}^2 + Q_{max}^2\mathbb{E}\bar{D}_{i+1,\lceil\frac{L}{T}\rceil-1}}{\mathbb{E}\bar{D}_{i,1}}\right) \\
&= \frac{\mathbb{E}D^{*3}}{3\mathbb{E}D^*} + \sum_{i=0}^{\infty}\frac{Q_{max}\left(\mathbb{E}\bar{D}_{i+1,\lceil\frac{L}{T}\rceil-1}^2 + Q_{max}\mathbb{E}\bar{D}_{i+1,\lceil\frac{L}{T}\rceil-1}\right)}{\mathbb{E}D}
\end{aligned}
$$

Appendix 9. (Numerical results of the reorder point calculation with the methods described in section 3)

		L=10						L=20					
		β=0.90			β=0.99			β=0.90			β=0.99		
T	Q_s	$c_D = 1$	$c_D = 2$	$c_D = 4$	$c_D = 1$	$c_D = 2$	$c_D = 4$	$c_D = 1$	$c_D = 2$	$c_D = 4$	$c_D = 1$	$c_D = 2$	$c_D = 4$
	∞	533 0.9003 543	769 0.9011 783	1652 0.9043 1672	930 0.9911 932	1469 0.9905 1469	3442 0.9896 3433	1115 0.9009 631	1404 0.8977 928	2381 0.8994 1926	1619 0.9910 1118	2299 0.9906 1801	4534 0.9913 4039
0.3L	25	422 0.8616 437	422 0.8454 437	421 0.8359 439	623 0.9650 625	623 0.9562 627	623 0.9478 620	942 0.8530 464	942 0.8334 465	942 0.8220 466	1195 0.9621 697	1196 0.9520 701	1195 0.9520 699
	50	446 0.8635 462	447 0.8244 467	448 0.7948 469	704 0.9720 705	708 0.9496 711	710 0.9323 714	985 0.8579 507	987 0.8134 514	987 0.7941 528	1312 0.9710 814	1316 0.9491 822	1319 0.9491 823
	100	483 0.8789 497	494 0.8195 517	499 0.7488 535	810 0.9834 811	839 0.9541 843	851 0.9115 858	1044 0.8759 563	1061 0.8168 598	1068 0.7496 620	1457 0.9832 959	1498 0.9549 1005	1514 0.9549 1023
	200	517 0.8946 531	572 0.8435 597	596 0.7070 640	899 0.9898 901	1033 0.9688 1033	1091 0.9043 1099	1096 0.8943 612	1175 0.8456 713	1211 0.7290 781	1577 0.9895 1078	1756 0.9697 1260	1834 0.9697 1354
	400	528 0.8991 540	670 0.8764 691	768 0.7288 820	928 0.9913 928	1266 0.9836 1268	1487 0.9175 1497	1112 0.8999 629	1309 0.8766 839	1441 0.7511 1014	1613 0.9911 1115	2055 0.9846 1560	2340 0.9846 1852
0.5L	25	422 0.8794 434	422 0.8756 434	421 0.8719 431	623 0.9780 623	623 0.9765 625	623 0.9750 624	942 0.8738 458	942 0.8731 463	942 0.8634 451	1195 0.9767 695	1196 0.9738 695	1195 0.9738 702
	50	446 0.8777 460	447 0.8645 463	448 0.8580 464	704 0.9810 706	708 0.9745 709	710 0.9731 715	985 0.8716 502	987 0.8586 506	987 0.8535 512	1312 0.9796 812	1316 0.9737 818	1319 0.9737 821
	100	483 0.8848 497	494 0.8562 513	499 0.8314 518	810 0.9856 810	839 0.9747 839	851 0.9670 854	1044 0.8829 562	1061 0.8537 589	1068 0.8337 606	1457 0.9863 960	1498 0.9753 1005	1514 0.9753 1008
	200	517 0.8948 530	572 0.8594 591	596 0.8115 623	899 0.9902 900	1033 0.9791 1035	1091 0.9605 1095	1096 0.8962 613	1175 0.8598 704	1211 0.8221 759	1577 0.9898 1076	1756 0.9797 1253	1834 0.9797 1338
	400	528 0.8993 540	670 0.8791 688	768 0.8069 802	928 0.9911 929	1266 0.9858 1267	1487 0.9629 1489	1112 0.9010 628	1309 0.8808 834	1441 0.8134 987	1613 0.9913 1114	2055 0.9864 1558	2340 0.9864 1846
L	25	422 0.8993 430	422 0.8999 430	421 0.9000 430	623 0.9908 623	623 0.9912 625	623 0.9908 624	942 0.8995 452	942 0.9010 453	942 0.8987 453	1195 0.9911 697	1196 0.9911 695	1195 0.9911 681
	50	446 0.8983 456	447 0.8994 458	448 0.8992 456	704 0.9912 705	708 0.9911 710	710 0.9914 713	985 0.8979 495	987 0.8991 500	987 0.8988 502	1312 0.9914 813	1316 0.9921 818	1319 0.9921 822
	100	483 0.8978 493	494 0.8991 506	499 0.8965 509	810 0.9916 809	839 0.9915 839	851 0.9915 848	1044 0.8984 557	1061 0.9001 578	1068 0.9051 584	1457 0.9912 957	1498 0.9927 1000	1514 0.9927 1017
	200	517 0.8999 531	572 0.9001 584	596 0.9000 611	899 0.9918 901	1033 0.9918 1032	1091 0.9920 1096	1096 0.8991 611	1175 0.8974 691	1211 0.9005 730	1577 0.9916 1077	1756 0.9920 1258	1834 0.9920 1331
	400	528 0.8988 541	670 0.8991 685	768 0.8978 787	928 0.9912 929	1266 0.9920 1266	1487 0.9925 1487	1112 0.8993 628	1309 0.9000 831	1441 0.9033 965	1613 0.9910 1114	2055 0.9919 1557	2340 0.9919 1839
1.5L	25	422 0.8989 429	422 0.8989 431	421 0.8986 429	623 0.9907 623	623 0.9910 625	623 0.9899 622	942 0.8992 453	942 0.9006 454	942 0.8948 448	1195 0.9905 696	1196 0.9899 693	1195 0.9899 706
	50	446 0.8988 456	447 0.8989 458	448 0.8999 460	704 0.9912 705	708 0.9915 710	710 0.9918 713	985 0.9004 498	987 0.8972 496	987 0.9010 501	1312 0.9913 813	1316 0.9912 818	1319 0.9912 815
	100	483 0.8983 494	494 0.8994 507	499 0.8965 509	810 0.9914 809	839 0.9918 840	851 0.9927 854	1044 0.8983 558	1061 0.9012 576	1068 0.9042 586	1457 0.9914 957	1498 0.9923 1002	1514 0.9923 1009
	200	517 0.8997 531	572 0.8991 585	596 0.8989 609	899 0.9919 901	1033 0.9915 1033	1091 0.9930 1088	1096 0.9006 613	1175 0.8974 691	1211 0.9004 727	1577 0.9916 1078	1756 0.9921 1260	1834 0.9921 1332
	400	528 0.8991 541	670 0.8979 685	768 0.8979 785	928 0.9911 928	1266 0.9917 1266	1487 0.9924 1488	1112 0.8991 629	1309 0.8989 830	1441 0.8993 961	1613 0.9912 1114	2055 0.9922 1558	2340 0.9922 1839

In each cel the top, the middle and the bottom elements denote the reorder point calculated by the method described in section 2, the associated actual service level and the average stock position, respectively.

Appendix 10. (Numerical results of the reorder point calculation with the methods described in section 4)

		L=10						L=20					
		β=0.90			β=0.99			β=0.90			β=0.99		
T	Q_s	$c_D = 1$	$c_D = 2$	$c_D = 4$	$c_D = 1$	$c_D = 2$	$c_D = 4$	$c_D = 1$	$c_D = 2$	$c_D = 4$	$c_D = 1$	$c_D = 2$	$c_D = 4$
	∞	533 0.9003 543	769 0.9011 783	1652 0.9043 1672	930 0.9911 932	1469 0.9905 1469	3442 0.9896 3433	1115 0.9009 631	1404 0.8977 928	2381 0.8994 1926	1619 0.9910 1118	2299 0.9906 1801	4534 0.9913 4039
	25	477 0.8981 486	500 0.8987 508	513 0.8983 522	766 0.9914 766	812 0.9916 813	835 0.9914 836	1025 0.9002 537	1056 0.9013 569	1072 0.8961 582	1389 0.9910 889	1448 0.9916 950	1480 0.9916 980
	50	504 0.8977 514	571 0.8980 580	621 0.8997 631	846 0.9911 846	981 0.9919 980	1067 0.9912 1065	1069 0.8991 583	1156 0.8986 668	1218 0.8960 728	1498 0.9911 998	1670 0.9920 1171	1780 0.9920 1284
$0.3L$	100	519 0.8984 531	653 0.8973 663	806 0.8974 813	897 0.9915 897	1180 0.9920 1180	1447 0.9928 1448	1096 0.8995 611	1268 0.8983 785	1450 0.9017 966	1570 0.9912 1071	1916 0.9918 1417	2250 0.9918 1753
	200	527 0.8981 538	712 0.8991 726	1075 0.9000 1088	922 0.9915 923	1340 0.9914 1341	2012 0.9927 2013	1109 0.9002 625	1349 0.8990 868	1766 0.9008 1286	1606 0.9912 1107	2117 0.9913 1616	2905 0.9913 2408
	400	528 0.8988 541	737 0.8976 752	1360 0.8952 1376	928 0.9912 929	1423 0.9911 1426	2691 0.9929 2691	1112 0.9001 630	1390 0.8977 914	2091 0.8972 1612	1614 0.9916 1115	2231 0.9913 1734	3634 0.9913 3135
	25	449 0.8984 457	454 0.8984 462	456 0.8991 466	697 0.9910 697	709 0.9908 709	714 0.9916 716	984 0.8981 496	991 0.8992 502	994 0.9016 508	1297 0.9908 798	1314 0.9915 815	1322 0.9915 823
	50	481 0.8992 491	502 0.8986 512	514 0.8992 524	788 0.9912 788	836 0.9911 836	861 0.9920 865	1035 0.8987 547	1065 0.8998 578	1082 0.8987 592	1422 0.9916 923	1488 0.9917 989	1521 0.9917 1026
$0.5L$	100	510 0.8991 521	576 0.8978 586	622 0.8986 633	871 0.9915 872	1012 0.9921 1011	1103 0.9916 1102	1081 0.8990 597	1171 0.9003 687	1232 0.8998 752	1535 0.9916 1036	1721 0.9917 1219	1840 0.9917 1339
	200	525 0.8983 538	662 0.8994 676	804 0.8998 818	919 0.9912 919	1220 0.9920 1221	1490 0.9917 1489	1106 0.9002 622	1288 0.8971 805	1466 0.8978 985	1599 0.9913 1100	1980 0.9914 1482	2328 0.9914 1837
	400	528 0.8988 541	724 0.8986 739	1072 0.8969 1087	928 0.9912 929	1391 0.9917 1392	2054 0.9918 2052	1112 0.9003 629	1373 0.8983 892	1785 0.8976 1312	1614 0.9909 1115	2183 0.9918 1687	2991 0.9918 2491
	25	422 0.8991 430	422 0.8983 429	422 0.9014 433	623 0.9906 623	623 0.9909 624	623 0.9909 623	942 0.8990 452	942 0.8991 452	942 0.9004 453	1195 0.9905 695	1196 0.9906 698	1196 0.9906 698
	50	446 0.8982 456	447 0.8990 457	448 0.8991 459	704 0.9913 705	708 0.9908 708	710 0.9916 710	985 0.8990 497	987 0.9006 500	987 0.9000 502	1312 0.9912 812	1316 0.9912 814	1319 0.9912 824
L	100	483 0.8987 494	494 0.8978 505	499 0.9008 512	810 0.9913 810	839 0.9920 840	851 0.9926 851	1044 0.8977 558	1061 0.8995 574	1068 0.9018 583	1457 0.9918 958	1498 0.9925 1000	1514 0.9925 1014
	200	517 0.8984 529	572 0.8987 585	596 0.8969 608	899 0.9915 899	1033 0.9920 1034	1091 0.9926 1092	1096 0.8996 612	1175 0.8992 692	1211 0.9012 728	1577 0.9912 1078	1756 0.9923 1256	1834 0.9923 1331
	400	528 0.8993 541	670 0.8984 686	768 0.8959 784	928 0.9909 928	1266 0.9914 1266	1487 0.9929 1487	1112 0.8994 628	1309 0.8991 828	1441 0.9009 964	1613 0.9915 1114	2055 0.9924 1556	2340 0.9924 1841
	25	422 0.8993 430	422 0.8986 429	422 0.9003 431	623 0.9906 623	623 0.9910 624	623 0.9908 622	942 0.8995 452	942 0.8992 453	942 0.9018 455	1195 0.9908 696	1196 0.9911 699	1196 0.9911 699
	50	446 0.8987 456	447 0.8990 458	448 0.8991 460	704 0.9910 705	708 0.9912 709	710 0.9908 707	985 0.9000 498	987 0.8994 498	987 0.8990 501	1312 0.9909 812	1316 0.9917 819	1319 0.9917 819
$1.5L$	100	483 0.8988 493	494 0.8980 506	499 0.8988 510	810 0.9913 810	839 0.9922 840	851 0.9920 853	1044 0.9000 559	1061 0.8991 575	1068 0.8991 585	1457 0.9915 958	1498 0.9913 998	1514 0.9913 1020
	200	517 0.8988 529	572 0.8975 584	596 0.8995 611	899 0.9916 900	1033 0.9921 1034	1091 0.9918 1092	1096 0.9002 611	1175 0.8984 693	1211 0.9025 733	1577 0.9914 1077	1756 0.9921 1255	1834 0.9921 1331
	400	528 0.8983 540	670 0.8972 685	768 0.8978 784	928 0.9912 928	1266 0.9916 1267	1487 0.9924 1486	1112 0.8996 629	1309 0.8993 830	1441 0.9008 964	1613 0.9911 1115	2055 0.9920 1556	2340 0.9920 1846

In each cel the top, the middle and the bottom elements denote the reorder point calculated by the method described in section 3, the associated actual service level and the average stock position, respectively.

Appendix 11. (Numerical results of the reorder point calculation with the methods described in section 5)

		L=10						L=20					
		β=0.90			β=0.99			β=0.90			β=0.99		
T	Q_s	$c_D = 1$	$c_D = 2$	$c_D = 4$	$c_D = 1$	$c_D = 2$	$c_D = 4$	$c_D = 1$	$c_D = 2$	$c_D = 4$	$c_D = 1$	$c_D = 2$	$c_D = 4$
	∞	533 0.9003 543	769 0.9011 783	1652 0.9043 1672	930 0.9911 932	1469 0.9905 1469	3442 0.9896 3433	1115 0.9009 631	1404 0.8977 928	2381 0.8994 1926	1619 0.9910 1118	2299 0.9906 1801	4534 0.9913 4039
$0.3L$	25	259 0.8986 453	142 0.8992 437	67 0.9001 424	496 0.9910 683	343 0.9907 631	226 0.9903 578	601 0.8988 485	370 0.8987 455	221 0.8993 431	897 0.9910 771	620 0.9912 698	419 0.9912 621
	50	403 0.8975 496	279 0.8988 492	132 0.8999 453	719 0.9912 804	582 0.9910 788	364 0.9901 678	873 0.8991 554	618 0.8994 540	339 0.8993 476	1269 0.9913 937	994 0.9911 906	625 0.9911 754
	100	492 0.8979 526	472 0.8982 598	285 0.8971 552	862 0.9911 887	923 0.9917 1038	680 0.9905 941	1041 0.8987 603	944 0.8995 688	583 0.8993 614	1506 0.9910 1054	1497 0.9915 1228	1058 0.9915 1079
	200	523 0.8981 538	644 0.8980 701	613 0.8953 805	918 0.9910 923	1242 0.9918 1287	1326 0.9922 1508	1102 0.8991 624	1226 0.9000 832	1051 0.8982 931	1599 0.9909 1105	1952 0.9916 1541	1878 0.9916 1743
	400	528 0.8983 541	723 0.8980 749	1090 0.8933 1199	928 0.9909 930	1402 0.9912 1414	2234 0.9922 2331	1112 0.8993 629	1365 0.8996 906	1680 0.8976 1393	1614 0.9907 1115	2196 0.9913 1716	3000 0.9913 2694
$0.5L$	25	186 0.8989 437	98 0.9000 427	49 0.9001 420	379 0.9908 623	261 0.9906 584	189 0.9903 555	458 0.8989 454	282 0.8995 436	186 0.8994 424	699 0.9909 687	484 0.9908 631	357 0.9908 589
	50	342 0.8979 476	205 0.8987 461	99 0.8994 439	616 0.9910 743	444 0.9910 694	283 0.9901 617	756 0.8990 519	481 0.8993 489	275 0.8995 451	1101 0.9911 853	778 0.9910 779	502 0.9910 671
	100	468 0.8975 516	377 0.8988 543	199 0.8994 497	818 0.9912 857	742 0.9912 900	488 0.9904 779	995 0.8992 586	780 0.8991 609	439 0.8990 531	1435 0.9913 1013	1231 0.9917 1048	791 0.9917 875
	200	520 0.8979 536	578 0.8985 658	437 0.8960 669	912 0.9912 918	1107 0.9918 1176	952 0.9919 1177	1095 0.8990 621	1118 0.8998 771	791 0.8985 751	1587 0.9910 1097	1765 0.9917 1405	1405 0.9917 1354
	400	528 0.8983 541	706 0.8979 736	867 0.8935 1012	928 0.9909 930	1366 0.9917 1383	1792 0.9931 1928	1112 0.8993 629	1334 0.8995 886	1377 0.8972 1164	1614 0.9907 1115	2136 0.9917 1669	2441 0.9917 2215
L	25	115 0.8998 424	60 0.9002 420	32 0.9002 418	267 0.9906 571	194 0.9904 550	157 0.9902 538	317 0.8995 432	207 0.8995 424	152 0.8997 420	505 0.9907 612	370 0.9906 582	301 0.9906 564
	50	261 0.8985 452	143 0.8995 440	75 0.8997 430	485 0.9912 670	330 0.9908 620	229 0.9904 580	601 0.8992 479	365 0.8993 454	230 0.8989 437	882 0.9912 750	597 0.9911 679	419 0.9911 621
	100	420 0.8975 497	276 0.8984 490	139 0.8989 462	735 0.9915 804	556 0.9916 764	352 0.9905 670	907 0.8993 555	605 0.8991 531	337 0.8989 480	1302 0.9914 938	953 0.9916 870	598 0.9916 734
	200	511 0.8980 532	476 0.8980 594	286 0.8976 557	894 0.9913 904	910 0.9923 1019	642 0.9914 908	1079 0.8990 614	949 0.8992 679	567 0.8982 607	1559 0.9911 1078	1481 0.9922 1200	992 0.9922 1023
	400	528 0.8983 541	665 0.8975 707	609 0.8968 799	928 0.9909 929	1272 0.9923 1301	1254 0.9932 1438	1112 0.8994 629	1266 0.8995 842	1027 0.8982 906	1614 0.9908 1115	2006 0.9922 1565	1785 0.9922 1654
$1.5L$	25	109 0.8998 419	56 0.9000 417	31 0.9002 417	260 0.9906 564	191 0.9904 546	155 0.9902 536	311 0.8994 426	203 0.8998 421	150 0.8998 419	498 0.9906 605	366 0.9906 577	299 0.9906 561
	50	248 0.8985 440	134 0.8990 431	72 0.8996 427	470 0.9910 655	318 0.9908 609	224 0.9903 575	588 0.8989 467	356 0.8993 446	226 0.8991 434	868 0.9912 737	586 0.9911 668	415 0.9911 616
	100	407 0.8977 485	252 0.8981 469	127 0.8994 451	718 0.9914 786	524 0.9912 732	332 0.9903 650	893 0.8994 543	582 0.8992 509	325 0.8990 469	1286 0.9914 922	924 0.9916 841	580 0.9916 716
	200	507 0.8983 528	433 0.8983 554	242 0.8984 515	887 0.9914 897	849 0.9918 957	567 0.9905 832	1075 0.8990 609	908 0.8988 641	523 0.8989 566	1552 0.9911 1072	1427 0.9921 1145	927 0.9921 959
	400	528 0.8983 541	629 0.8971 673	484 0.8981 679	927 0.9909 929	1211 0.9921 1241	1045 0.9917 1231	1111 0.8990 628	1233 0.8993 812	910 0.8985 795	1613 0.9908 1114	1956 0.9921 1516	1607 0.9921 1477

In each cel the top, the middle and the bottom elements denote the reorder point calculated by the method described in section 4, the associated actual service level and the average stock position, respectively.

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